USING BARYCENTRIC COORDINATES FOR SERENDIPITY SURFACE MODELING

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***Summary.* In work there is the probabilistic-geometrical method of design of bases of three-cornered eventual elements with the use of barycentric coordinates is widespread on the design of bases of serendipity eventual elements.**

***Keywords:* an eventual element, base function, geometrical probability, barycentric coordinate.**

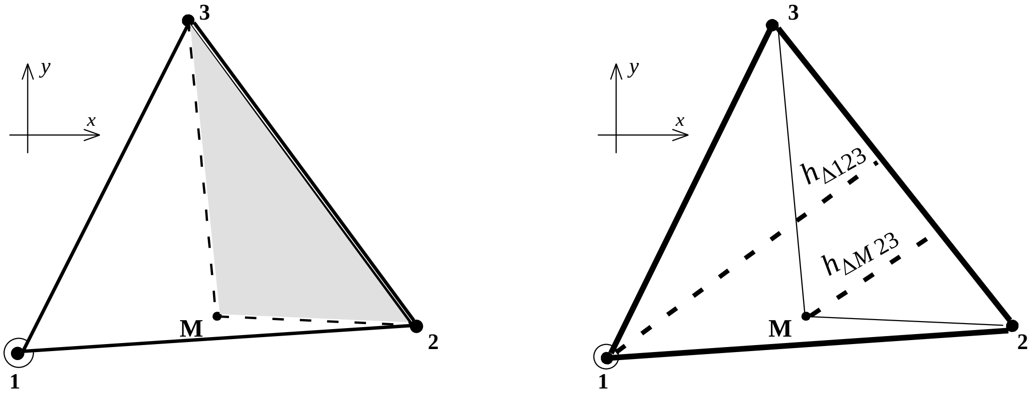
Formulation of the problem. The traditional method of construction of interpolation polynomial for an eventual element (EE) needs a stowage and decision of SLAE in relation to parameters that determine a polynomial [1]. Than higher degree of polynomial, that realization of method of determination of base functions becomes more difficult. There are cases (for example, for THIS in form hexagon), when it is impossible to build a base matrix methods THIS through the singular matrices of the system. Therefore for the construction of interpolation polynomials it comfortably to use other methods. An alternative to the matrix method is a method that is based on probabilistic-geometrical presentations. In monographs from the method of eventual elements (MEE) barycentric coordinates are widely used for the construction of bases on three-cornered and tetrahedral elements [1-2]. Unfortunately, the founders of MEE did not use this approach for eventual elements in form square and cube.

Analysis of recent researches. As known, natural generalization and expansion of concept of classic probability on the unfinished great number of points is geometrical probability that is calculated as a relation of measures (lengths, areas, volumes) in one-, two- and three-dimensional cases. First for the construction of base of eventual element geometrical probability was used by A.N. Homchenko in 1982 [3]. Later by means of probabilistic-geometrical method there were the got bases on three-cornered [3,4,6] and quadrangular eventual elements [5,6]. The basic idea of construction of base function for a certain knot is formulated as a task of being of probability of entry of the arbitrary point thrown in at random in an eventual element in a subregion that is opposite to this knot. On this time authors are work out a few going near the probabilistic-geometrical design of bases quadrangular EE [5,6]. One of these approaches is based on partition quadrangular EE the special methods on triangles and application in last barycentric coordinates.

*The wording of the purposes of the article.*To apply the probabilistic-geometrical method of design of bases of three-cornered eventual elements with the use of barycentric coordinates to the design of bases of serendipity of eventual elements.

Main part. In 1827 the new barycentric system of coordinates offered by the German geometer and astronomer August Ferdinand Mjobius. This system of coordinates in two-dimensional case is considered set, if a base triangle is set. The barycentric coordinates (ξ_1, ξ_2, ξ_3) of any point M of plane are determined as parts of single mass, that it follows to place in the corresponding tops of base triangle, that a point M became the barycenter of triangle. The barycentric coordinate of simplex can be calculated both relation of areas of triangles (fig. 1, a) and as a relation of heights of corresponding triangles (fig. 1, б).

Base function for a knot 1 three-cornered eventual element too can be calculated by means of formula of geometrical probability as relation of areas (or heights) of corresponding triangles (fig. 1).



a) б)

Fig. 1. Determination of barycentric coordinates in the eventual element of linear interpolation.

For example, the first coordinate is set by a formula:

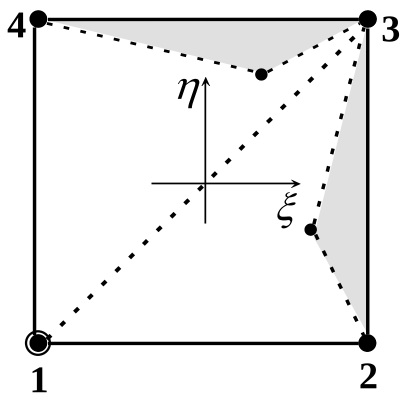
$$\xi_1(x,y) = \frac{S_{\Delta M23}}{S_{\Delta 123}} = N_1 \quad \text{або} \quad \xi_1(x,y) = \frac{h_{\Delta M23}}{h_{\Delta 123}} = N_1, \tag{1}$$

where $S_{\Delta 123}$ - an area of triangle 1-2-3; $S_{\Delta M23}$ − an area of triangle M-2-3; $M(x,y)$ − current point, $h_{\Delta M23}$− height of triangle M-2-3, $h_{\Delta 123}$− height of triangle 1-2-3.

Thus, from a formula (1) becomes clear that barycentric coordinate of simplex and base function three-cornered EE with three knots of − this concept identical. Will notice that surface of base function three-cornered EE with three knots on the border of − it is a plane that has applicate even of unit in a corresponding knot and zero in other knots. The sum of base functions on EE equals unit for the arbitrary point of triangle. Behavior of surfaces of base functions of simplex characterizes properties of barycentric coordinates brightly.

Will show, as barycentric coordinates of simplex can be used on serendipity EE. Serendipit element is a quadrangle with the knots of interpolation only on the border of element and base that answers these knots. The elements of serendipity family allow due to absence of internal knots considerably to shorten the volume of calculations in comparing to the elements of lagrange as, to remove unphysical oscillations of the field. Serendipit elements were open in 60th past century, and since then interest grows unceasingly in them, that is accompanied by distribution of them practical application.

Bilinear EE has four knots in the tops of square (SEE- 4) (fig. 2). Will present SEE- 4 as composition of two triangles with a top in a general knot 1: the first triangle of Δ_{1-2-3}–, the second triangle of – Δ_{1-3-4} (fig. 2).



$|\xi| \le 1,$

$|\eta| \le 1.$

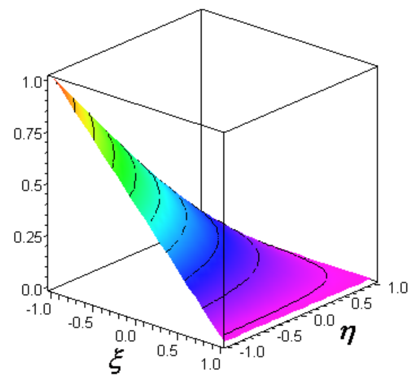


Fig. 2. To the construction of function N_1.

Fig. 3. Visualization of N_1

On each of triangles build the base of linear interpolation after a formula (1) and, taking advantage of theorem of product of probabilities of independent events, get a base function for a knot 1:

$$N_1 = N_1^{(1)} \cdot N_1^{(2)} = \frac{1-\xi}{2} \cdot \frac{1-\eta}{2} = \frac{1}{4}(1-\xi)(1-\eta). \qquad (2)$$

In general,

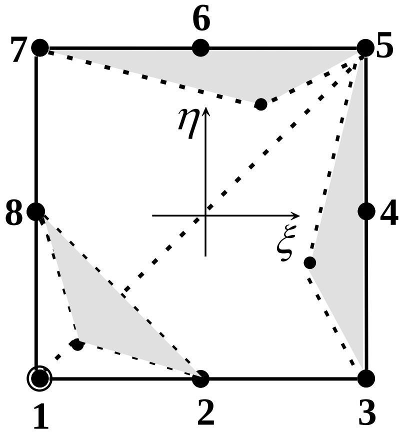
$$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta), \quad i = 1,2,3,4, \quad \xi_i, \eta_i = \pm 1. \qquad (3)$$

Visualization of serendipity surface N_1 (fig. 3) demonstrates, that functions (3) satisfy to basic properties of base functions.

THIS opens family of serendipity EE higher orders bisquare element that has 8 knots (SEE- 8). On this element it is too possible to take advantage of probabilistic-geometrical method. For the construction of base function N_1 break up SEE- 8 on three triangle: Δ_{1-3-5}, Δ_{1-5-7} and

Δ_{1-2-8} (fig. 4). On each of triangles build the base of linear interpolation after a formula (1):

$$N_1^{(1)} = \frac{1-\xi}{2}; \quad N_1^{(2)} = \frac{1-\eta}{2}; \quad N_1^{(3)} = -1-\xi-\eta. \qquad (4)$$



$$|\xi| \le 1, \quad |\eta| \le 1.$$

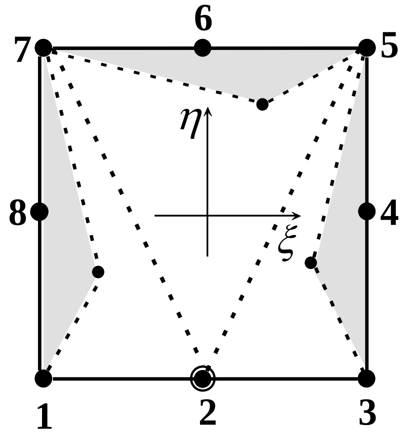


Fig. 4. To the construction N_1 To the base SEE-8.

Fig. 5. To the construction N_2 to the base SEE-8.

The product of these probabilities gives a base function for a knot 1:

$$N_1 = N_1^{(1)} \cdot N_1^{(2)} \cdot N_1^{(3)} = \frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta). \qquad (5)$$

For the construction of base function for a knot 2 break up a bisquare element on three triangle: 1) Δ_{2-3-5}, 2) Δ_{2-7-1} and 3) Δ_{2-5-7} and on each of them search a corresponding barycentric coordinate $N_2^{(1)}, N_2^{(2)}, N_2^{(3)}$ (fig. 5) :

$$N_2^{(1)} = 1-\xi; \quad N_2^{(2)} = 1+\xi; \quad N_2^{(3)} = \frac{1-\eta}{2}. \qquad (6)$$

After the theorem of product of probabilities get:

$$N_2 = N_2^{(1)} \cdot N_2^{(2)} \cdot N_2^{(3)} = \frac{1}{2}\left(1-\xi^2\right)(1-\eta). \qquad (7)$$

Other base functions of base of bisquare interpolation will get from formulas (5) and (7) transposition of coordinates ξ and η .

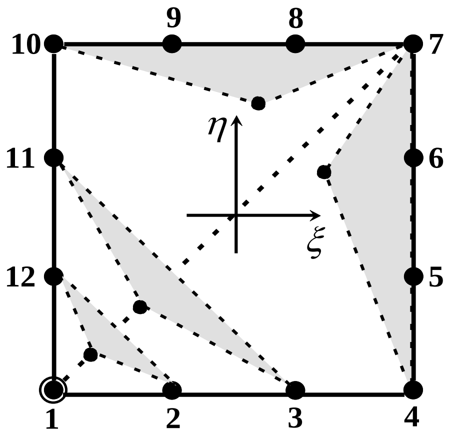
Bicube an eventual element has 12 knots of interpolation. First polinomial base for this purpose EE was got a selection in 1968 Ergatudis, Airons and Zenkevich [1]. Authors are get alternative bases on this element [4]. Will show, as possible to use barycentric coordinates for the construction of alternative base of SEE- 12 EE.

For the construction of base function N_1 break up EE on four triangle: $\Delta_{1-4-7}, \Delta_{1-7-10}$, Δ_{1-3-11} and Δ_{1-2-12} with a general top 1 (fig. 6). On each of triangles build the base of linear interpolation after a formula (1):

$$N_1^{(1)} = \frac{1}{2}(1-\xi); \qquad N_1^{(2)} = \frac{1}{2}(1-\eta);$$
$$N_1^{(3)} = \frac{1}{4}(-2-3\xi-3\eta), \quad N_1^{(4)} = \frac{1}{4}(-4-3\xi-3\eta). \tag{8}$$

After the formula of product of probabilities get:

$$N_1 = N_1^{(1)} \cdot N_1^{(2)} \cdot N_1^{(3)} \cdot N_1^{(4)} =$$
$$= \frac{1}{32}(1-\xi)(1-\eta)(2+3\xi+3\eta)(4+3\xi+3\eta). \tag{9}$$



$$|\xi| \le 1, \quad |\eta| \le 1.$$

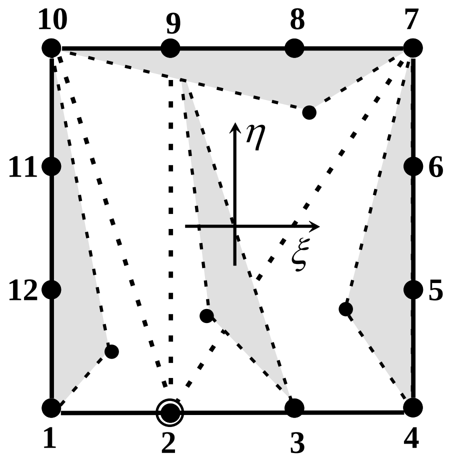


Fig. 6. To the construction N_1 To the base SEE-12.

Fig. 7. To the construction N_2 To the base SEE-12.

For a construction N_2 will consider triangles: $\Delta_{2-4-7}, \Delta_{2-10-1}$ and Δ_{2-7-10} (fig.7). Get corresponding probabilities:

$$N_2^{(1)} = \frac{3}{4}(1-\xi), \qquad N_2^{(2)} = \frac{3}{2}(1+\xi),$$
$$N_2^{(3)} = \frac{1}{2}(-3\xi-\eta), \qquad N_2^{(4)} = \frac{1}{2}(1-\eta). \tag{10}$$

By rule of product of probabilities get:

$$N_2 = N_2^{(1)} \cdot N_2^{(2)} \cdot N_2^{(3)} \cdot N_2^{(4)} = \frac{9}{32}\left(1-\xi^2\right)(1-\eta)(-3\xi-\eta). \tag{11}$$

Conclusions. Possibility of application of probabilistic-geometrical method (with the use of barycentric coordinates) is first shown to the construction of equalization of interpolation serendipity surface. The prospects of further researches logically to bind to generalization of the investigated method on eventual voxels.

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