THE THEOREM OF CLAIRAUT FOR CONSTRUCTION OF GEODETIC LINES AT FUNCTION OF LENGTH OF THE ARC ON SURFACE OF REVOLUTIONS

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Summary. **On the basis of the Clairaut's theorem we received new dependencies to build geodesic lines on the surface of rotation. The independent variable is the arc length of these lines. The authenticity of the results is checked on the cone. To construct of the geodesics in different directions conical reamer was used, in which these lines are straight.**

***Keywords*: geodetic line, surface rotation, Clairaut theorem, the arc length, the system of differential equations.**

Formulation of the problem. Search geodesic lines on the surface of the whole is reduced to drafting of the second-order differential equations, for integration to which you need to apply numerical methods. Only certain surfaces (surfaces of surface Liouville rotation) the order differential equation can be reduced to the first and expressions reduced to integration [1], which rarely again can be integrated. For surfaces of revolution following expression for integrating expression is based on Clairaut theorem, which establishes the relationship between the inner surface coordinates in the form $v = v(u)$, where v - angle point of the geodesic line around the axis of rotation of the surface. However, in this case the geodetic line full build is not always possible, as when changing parameter u, which is independent of the construction of the geodesic line, gradually covered the entire surface, for example, from top to bottom. But in the lower or upper surface of the geodesic line is given no direction, it touches at a certain point to some parallels and returns to the opposite side. The independent variable u cannot at some point start to decline, it only increases monotonically. Thus, according the Clairaut theorem can be built only fragments of geodesic lines.

Analysis of recent research. If the surface of the rotation to produce composite materials, reinforced threads, the thread should be reeled along these geodesic lines [2,3]. In addition, during the construction of some working bodies (including tillage), take into account the location of the geodesic lines on the surface, because the particles of the process material at forced their movement on working body trying to move along these lines close to surveying, especially at high speeds of movement [4 5]. Given the

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practical importance of geodesic lines, their location and construction was studied by different researchers [6,7].

The wording of the purposes of the article. Write a differential equation of geodesic lines in length function of their own arc for surfaces of rotation.

Main part. If the rotation surface is established by parametric equations in the form:

$$X = \varphi \cos v; \qquad Y = \varphi \sin v; \qquad Z = \psi, \tag{1}$$

where $\varphi=\varphi(u)$; $\psi=\psi(u)$ – parametric equation of meridian, according Clairaut theorem the internal equation of geodesic line is in the form $v = v(u)$ is described the integral:

$$v = c\int \frac{\sqrt{\varphi'^2 + \psi'^2}}{\varphi\sqrt{\varphi^2 - c^2}} du, \tag{2}$$

where c- constant, which determines the direction of the geodesic line to a specific point in the set values of internal coordinates u and v.

Expression (2) is connecting the inner surface coordinate with dependence $v = v(u)$. However, the relationship between internal coordinates can be set differently - with a new independent variable - length of the arc of the geodesic line, ie in the form $v=v(s)$ and $u = u(s)$. After differentiating expression (2) becomes:

$$\frac{dv}{du} = \frac{c}{\varphi}\sqrt{\frac{\varphi'^2 + \psi'^2}{\varphi^2 - c^2}}. \tag{3}$$

Since $v = v(s)$ and $p = u(s)$, you can record $\frac{dv}{du} = \frac{dv}{ds} : \frac{du}{ds} = \frac{v'}{u'}$. Substituting the obtained expression in (3), we obtain:

$$v' = \frac{u'c}{\varphi}\sqrt{\frac{\varphi_u'^2 + \psi_u'^2}{\varphi^2 - c^2}}. \tag{4}$$

It is understood that the expression (4) v 'and u' are derived in the variable s, and φ 'and ψ' - the variable u, what that variable is used in the lower index. The differential equation (4) cannot be used because it includes two unknown functions: $v = v(s)$ and $u = u(s)$. So for them of the need to have another equation. These equations can know identities for the curve, which is given in arc length function s: $x'^2 + y'^2 + z'^2 = 1$. Let's find the first derivatives of equations (1) with help of the variable s (with capital letters in equations (1) to replace small letters because in the first case the equation describing the surface, and the second - line on it):

$$\begin{aligned} x' &= u'\varphi'_u \cos v - v'\varphi \sin v; \\ y' &= u'\varphi'_u \sin v + v'\varphi \cos v; \\ z' &= u'\psi'_u. \end{aligned} \tag{5}$$

After substituting in the above identities of derivatives (5) we get:

$$v'^2\varphi^2 + u'^2\left(\varphi_u'^2 + \psi_u'^2\right) = 1. \quad (6)$$

Let's solve (6) due to u'':

$$u' = \sqrt{\frac{1 - v'^2\varphi^2}{\varphi_u'^2 + \psi_u'^2}}. \quad (7)$$

Now we have two equations (4) and (7) with two unknown functions. Substituting (7) (4) and after simplification we will obtain a simple relationship:

$$v' = \frac{dv}{ds} = \frac{c}{\varphi^2}. \quad (8)$$

After substituting (8) into (7) we have:

$$u' = \frac{du}{ds} = \frac{1}{\varphi}\sqrt{\frac{\varphi^2 - c^2}{\varphi_u'^2 + \psi_u'^2}}. \quad (9)$$

After separation of variables in (9) we finally obtain:

$$s = \int \varphi \sqrt{\frac{\varphi_u'^2 + \psi_u'^2}{\varphi^2 - c^2}} du. \quad (10)$$

By the integral (10) are expressions that define meridian, and their derivatives, which are variable functions u. If it can integrate, we get dependency $s=s(u)$. The next step - from the resulting inverse dependence found $u=u(s)$. You must dependency $u=u(s)$ substitute the function $\varphi(u)$, then get $\varphi(s)$. And the last stage - the dependence of the $v=v(s)$ the formula (8), which also can be written as an integral:

$$v = c\int \frac{ds}{\varphi^2[u(s)]}. \quad (11)$$

Expressions rarely integrate. They can be used to build geodesic lines numerical methods. Analytical expressions in the final form can be obtained only for the simplest surfaces of revolution.

Consider the following example. Take cone meridian parametric equations which are as follows [8]:

$$\varphi = e^{u\cos\beta}; \qquad \psi = e^{u\cos\beta}\mathrm{tg}\beta, \quad (12)$$

where β – angle rectilinear generators of the cone to the base.

We write the original equation (12):

$$\varphi_u' = e^{u\cos\beta}\cos\beta; \qquad \psi_u' = e^{u\cos\beta}\sin\beta. \quad (13)$$

Substitute (12) (13) (10) and get:

$$s = \int \frac{e^{2u\cos\beta}ds}{\sqrt{e^{2u\cos\beta} - c^2}} = \frac{1}{\cos\beta}\sqrt{e^{2u\cos\beta} - c^2} + c_1, \quad (14)$$

Where c_1 – constant of integration.

Solves (14) with respect to *u* (where $c_1=0$):

$$u=\frac{1}{\cos\beta}\log\sqrt{c^2+s^2\cos^2\beta}\,. \tag{15}$$

Substitute (15) in the first expression (12) and obtain dependence *φ(s)*: $\varphi=\sqrt{c^2+s^2\cos^2\beta}$. According to the formula (11) we find the dependence *v=v(s)*:

$$v=c\int\frac{ds}{c^2+s^2\cos^2\beta}=\frac{1}{\cos\beta}\mathrm{arctg}\left(\frac{s\cos\beta}{c}\right). \tag{16}$$

Equation (15), (16) are internal equation of geodesic lines on the cone, the direction of which depends on sustainable *c*. To check the reliability of the results, we use the known formula (2):

$$v=c\int\frac{du}{\sqrt{e^{2u\cos\beta}-c^2}}=\frac{1}{\cos\beta}\mathrm{arctg}\left(\frac{\sqrt{e^{2u\cos\beta}-c^2}}{c}\right). \tag{17}$$

If (1) substitute meridian equation (12) and dependence *v=v(u)* with (17), we obtain the parametric equations of the geodesic line. Its length s can be found in well-known formula integrating the square root of the sum of squares of derivatives parametric equations. In general terms, at *v=v(u)* This formula takes the form:

$$s=\int\sqrt{\varphi_u'^2+\psi_u'^2+\varphi^2 v_u'^2}\,du\,. \tag{18}$$

If (18) to substitute the expression (12), (13) and the derivative v 'from (17) (integrand expression), we get the exact same integral as (14). This suggests that the formula (10), (11) are correct.

When the requirement that all geodetic line went from one point in different directions depending on the constant *c* at *s=0*, find the constant of integration in c_1 (14) for a given initial coordinate u_0 on the surface of the cone:

$$c_1=-\frac{1}{\cos\beta}\sqrt{e^{2u_0\cos\beta}-c^2}\,. \tag{19}$$

Then the expression (15) becomes quite cumbersome form:

$$u=\frac{1}{\cos\beta}\log\sqrt{c^2+\left(s\cos\beta+\sqrt{e^{2u_0\cos\beta}-c^2}\right)^2}\,. \tag{20}$$

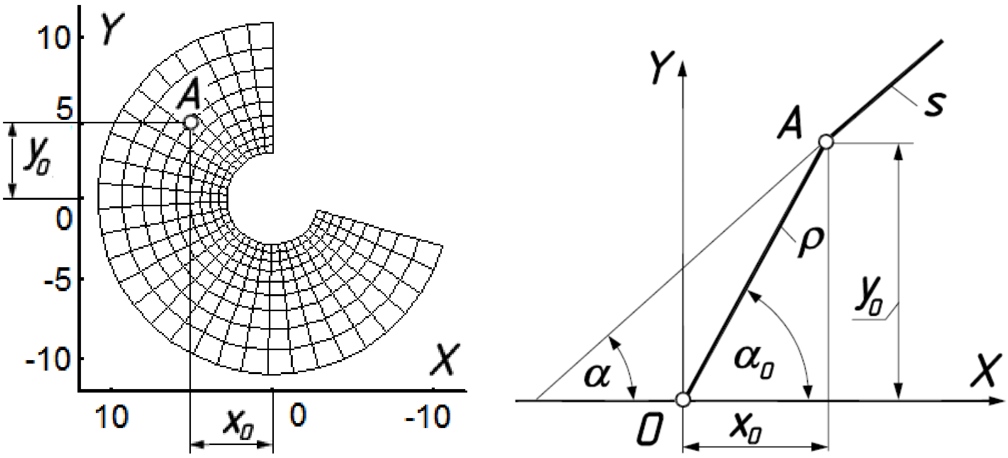
Analyzing the expression (20), we can conclude that the internal expression basal is found on sustainable difference. This means that the constant *c* restricted, that in not every direction you can build a geodesic line.

Thus, the construction of geodesic lines on the dependence based on Clairaut theorem cannot be implemented in full. However, this obstacle can be overcome for evolvent surfaces, of parametric equation is known. Geodesic lines on the evolvent converted to straight. So, we can build in inverse order to the direct beam, which on the surface are converted to geodesic lines emanating from a given point in different directions. Consider this cone example.

Parametric equations cone evolvent with the meridian of the form (12) are as follows:

$$X_p = \frac{e^{u\cos\beta}}{\cos\beta}\cos(v\cos\beta); \qquad Y_p = \frac{e^{u\cos\beta}}{\cos\beta}\sin(v\cos\beta)\ . \qquad (21)$$

If you find the first evolvent quadratic form (21) and cone (1) with the meridian (12), you can make sure they are the same. Sketch. 1, and the equations (21) built in the involute cone $\beta=45^0$. Let point A with coordinates x_0 and y_0 need to draw a straight line, which will be a geodesic on the cone. Consider this point in the coordinate system *OXY* separately (Fig. 1b).



a b

Fig.1. Before the definition of point A on cone evolvent , because we need to draw a straight line with a given direction:

a) point *A* on the evolvent with given coordinates x_0=5 and y_0=5;

b) determining the coordinates x_0 and y_0 through distance ρ and angle α_0 and direction of the straight line *s* through the angle α.

Parametric equation of the line *s* (and *s* - is the length of the line - independent variable) is written:

$$x = x_0 + s\cos\alpha; \qquad y = y_0 + s\sin\alpha\ , \qquad (22)$$

where α - the angle which sets the direction of the line.

Taking into account that $x_0 = \rho\cos\alpha_0$; $y_0 = \rho\sin\alpha_0$ line equation (22) is written:

$$x = \rho\cos\alpha_0 + s\cos\alpha; \qquad y = \rho\sin\alpha_0 + s\sin\alpha\ . \qquad (23)$$

To build straight (23) on the evolvent cone (21), needs to find its internal equation in the form *v = v (s)* and *u = u (s)*. For this among themselves liken corresponding coordinates of equations (21) and (23):

$$\rho\cos\alpha_0 + s\cos\alpha = \frac{e^{u\cos\beta}}{\cos\beta}\cos(v\cos\beta);$$
$$\rho\sin\alpha_0 + s\sin\alpha = \frac{e^{u\cos\beta}}{\cos\beta}\sin(v\cos\beta). \qquad (24)$$

Solve the system of equations (24) in connection with *v* and *u:*

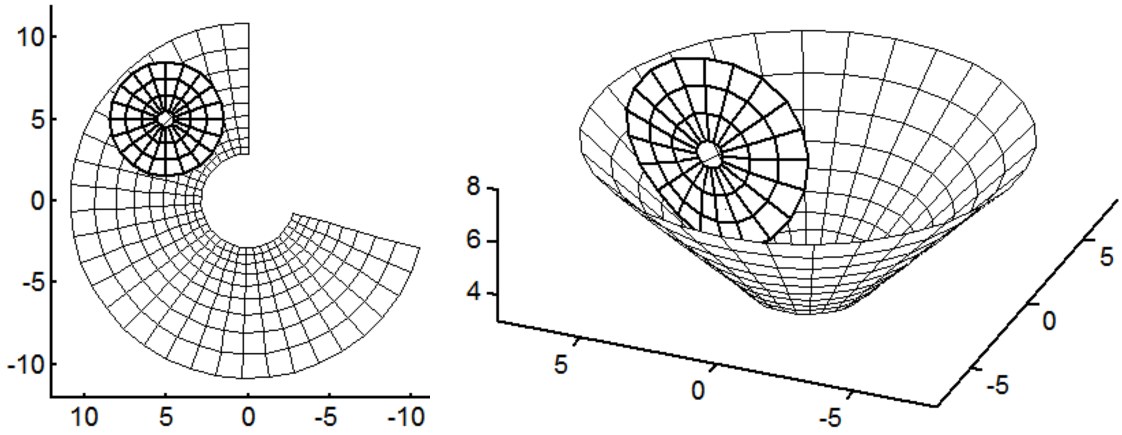
$$v = \frac{1}{\cos\beta}\operatorname{arctg}\left(\frac{\rho\sin\alpha_0 + s\sin\alpha}{\rho\cos\alpha_0 + s\cos\alpha}\right);$$
$$u = \frac{1}{\cos\beta}\ln\left(\cos\beta\sqrt{\rho^2 + s^2 + 2\rho s\cos(\alpha-\alpha_0)}\right). \qquad (25)$$

When substituting expressions (25) for given values of constants β ρ, α_0, α evolvent in equation (21) we obtain the corresponding straight line towards the curvilinear coordinates of the evolvent. At the same substitution of internal equations parametric equations, we obtain the appropriate geodesic lines of cone on its surface.

In constructing straight lines on the cone evolvent with a different angle α and s during the change from zero to a given value, end segments will lie on the circle. If in the expression (25) we make the angle α the second independent variable, then substituted in (21) we get a second orthogonal grid, one family of coordinate lines, which are straight, and the second - concentric circles (Fig. 2a). On the surface of the cone straits directly convert to geodetic line, and the circle - in the curves of constant geodesic curvature. This follows from the fact that the geodesic curvature of the curve does not change when bent surface and becomes a full evolvent on curvature, that is, in the curvature circles for our case. Such a system of coordinates on the surface is called half-developable [1]. Curves family of constant geodesic curvature (in our case - the family of concentric circles) called Darboux circles. In general, they can be unlocked (eg screw line on the surface evolvent helicoid).

When $\rho = 0$ in equations (25) are bunch of straight geodetics of cone generating and circles Darboux - its parallels.

Conclusions. The obtained integrals (10) and (11) are a modification of Clairaut theorem for finding the geodesic lines on the surfaces in rotation length function of the of the arc according to their own known equations of meridian. They can build a bunch of geodesic lines from a given point on the surface in different directions with the same length. Because of the specific integrals the construction of lines in some areas can be more difficult. This obstacle can be overcome for the evolvent surface if known parametric equation of evolvent. For this inverted order is cluster of straight lines and concentric circles scan, which is corresponded a bunch of geodesic lines and circles on the surface of Darboux.



a b

Fig.2. Cone bunch of geodesic lines and Darboux circles on it:
a) evolvent cone; b) cone in axonometry.

Literature

1. Погорелов А.В. Дифференциальная геометрия / А.В. Погорелов – М.: «НАУКА», 1969. – 176 с.
2. Завидский А. В. Определение параметров технологической поверхности, обеспечивающей непрерывность намотки по геодезическим линиям / А.В. Завидский // Труды МАИ. 1976. - № 349. - С. 34-35.
3. Якунин В.И. Вопросы геометрического проектирования процесса намотки составной поверхности / В.И. Якунин, В.А. Калинин, Т.В. Аюшев // Компьютерная геометрия и графика в инженерном образовании: Материалы всесоюзной конференции. - Нижний Новгород, 1991. - С. 149.
4. Войтюк Д.Г. Побудова геодезичних ліній, як граничних траєкторій руху матеріальних частинок по поверхні / Д.Г. Войтюк, С.Ф. Пилипака // Науковий вісник Національного аграрного університету. –К.: НАУ, 2003. –Вип. 60. –С. 138-141.
5. Юрчук В.П. Проектування поверхні роторного копача шляхом використання геодезичної лінії / В.П. Юрчук, О.Г. Гетьман // Труды Таврической государственной агротехнической академии. Вып.4. Прикл. геометрия и инж. графика. - Т.6. – Мелитополь: ТГАТА, 1999. – С. 85 – 88.
6. Пилипака С.Ф. Дослідження геодезичних ліній на поверхні гвинтового коноїда / С.Ф. Пилипака, Т.В. Гнітецька // Сучасні проблеми геометричного моделювання. Матеріали міжнародної науково-практичної конференції. – Львів: Національний університет "Львівська політехніка", 2003. –С. 77 - 80.

7. Урмаев Н.А. Приведенная длина геодезической линии / Н.А. Урмаев // Известия АН СССР. - Сер. матем., 5:4-5 (1941). – С. 369–376.
8. Кремець Т.С. Конформне відображення написів на ізометричні сітки конуса та кулі / Т.С. Кремець // Технічна естетика і дизайн. – К.: Віпол, 2011. – Вип. 9. – С. 112 – 117.