DETERMINATION CRITICAL VALUES OF PARAMETERS DIFFERENTIAL EQUALIZATIONS OF VIBRATIONS THROUGH CURVATURES OF PHASE TRAJECTORIES

L. Kutsenko, O. Semkiv

***Summary.* A graphic analytical method over of determination of critical values of parameters of phase trajectories of differential equalizations is Brought the second order, that is based on the concept of distorted of phase trajectories and takes into account the change of sign of their curvature along trajectories.**

***Keywords:* phase trajectory, critical value, the analysis of the qualitative level, bending of curve, curvature curve.**

Formulation of the problem. The traditional object of study is an analysis of theoretical mechanic's pendulum fluctuations. Their study on a qualitative level, is conveniently carried out by phase trajectories. The essence of this method is to describe the behavior of fluctuations using visual geometric images - phase portraits [1,2] constructed on the surface in rectangular coordinates with the axles of "bias" and "speed". In the specification, there may be fluctuations in at least one parameter which is substantially affect the nature of the fluctuations. For example, for the simple pendulum in such a parameter can serve as the initial velocity of its movement. Depending on its size will either be damped fluctuations, or "rotating" around the point of suspension. That is changing certain velocity values can distinguish between the "quality" of fluctuation. Those settings in which change the qualitative or topological properties of movement, called critical or bifurcation values [2, 3, 5]. For practical productions necessary engineering methods for calculating the critical values of the parameter fluctuations, the account of which can improve the design or oscillatory system, or prevent its alarm state [6, 7]. This points to the relevance of research topics.

Analysis of recent research. Search of critical values of the parameters of the phase trajectories (bifurcation theory) laid by A. Puankare and A. Lyapunov, then these studies were developed by O. Andronov and students [1, 2]. This work [3] shows provides an overview of different ways to study the phase trajectories on a qualitative level. But among them was only a few pure graphics that rely on geometric interpretation isoclines like solving equations. This indicates a lack of development grapho-analytical ways to find the critical values of the parameters of the phase trajectories for engineering practice. Ways to find the critical values of the parameters of the phase trajectories would be appropriate to supplement and those based on the nature of the curvature of the phase trajectory, and which are determined by a set of values of curvature along this path.

The wording of the purposes of the article. Development of the method of determining the critical parameters of the phase trajectories of the second order differential equations, based on the notion of curvature of the trajectory, and takes into account the change in the sign of the curvature along trajectories.

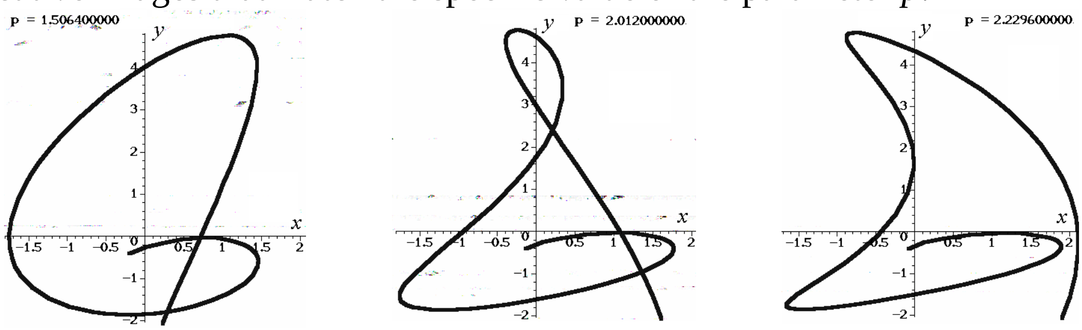
Main part. We assume that the motion of a point along the phase trajectory is carried out within the limits that define the boundaries of the parameter t changes of time, and "turn" right or left when driving is defined by various characters in the value of the curvature of the trajectory.

1. *The essence of the method.* To explain the essence of the method instead of the first phase trajectories consider a family of curves, in which the curvature of the elements can be calculated exactly:

$$x = \sin(pt) + a\cos(t)/2; \quad y = -t\sin(t), \tag{1}$$

where the parameter t varies within $t_{MIN} = -0{,}2\pi < t < t_{MAX} = 2{,}1\pi$, and the control parameter p varies within $p_{MIN} = 1{,}5 < p < p_{MAX} = 2{,}3$.

You must define a critical value at which items curves change family curvature - that must change in *quality level*. Construct a series of consecutive images that match the specific value of the parameter *p*.



p = 1.51 p = 2.01 p = 2.23

Fig.1. Image depending on the setting *p*

Analyzing the figures (preferably in computer animation mode) is easy to see that the family of curves can be divided into three components curvature its elements to be separated the two curves corresponding to the value of the parameter p = 1.7 and p = 2.15. The method of determining the critical control parameter values that does not rely on the animated image elements of the family of curves. That is the way to solve this problem on a formal level.

The method of the definition of critical control parameter values is proposed, which would not rely on the animated image elements of the family of curves. That is the way to solve this problem on the formal level.

It is based on the two theses.

Thesis 1. Critical values of control parameter family of curves correspond to the change in their elements in the quality level.

Thesis 2. Qualitative change of the family of curves elements can be analyzed to track the changing nature of the curvature values by using functions curvature lines.

Referring to [4], we calculate the curvature function for the family (1):

$$k(p) = \frac{u(-2\cos(t) + t\sin(t)) - v(-p^2\sin(pt) - p\cos(t)/2)}{(u^2+v^2)^{\frac{3}{2}}}, \qquad (2)$$

where $u = p\cos(pt) - p\sin(t)/2$ and $v = -\sin(t) - t\cos(t)$.

Sketch. 2 shows the element of curves family and the corresponding graph *k (t)* of curvature for the value *p* = 2 (this graph curvature *k (t)* is limited by direct k = 6):

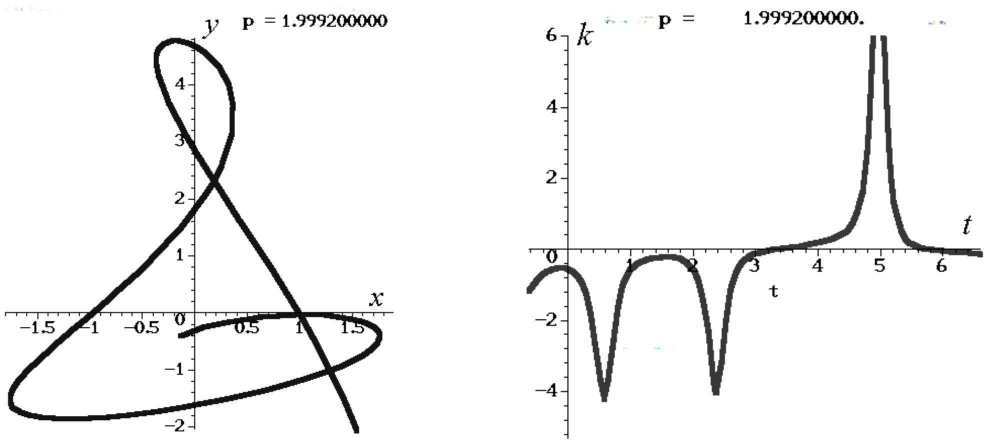


Fig. 2. Item of family curves and graphic of curvature *k (t)* for the value *p = 2.*

Next, examine change of the graphic function *k = k (t)* depending on the parameter *p* on the interval [1.5; 2,3]. S area between the graph and the abscissa is some constant number. However, you can view variables too - the area under the graph of k (t) depending on the parameter p. Then the area is not constant, but a function of *p: S = S (p).*

The program was compiled to determine the function *S (p)* as integral of curvature *k (t),* calculated within the parameter *t*. As a result, we obtain the graph of *S (p)* (Fig. 3, right - the same was increased, too).

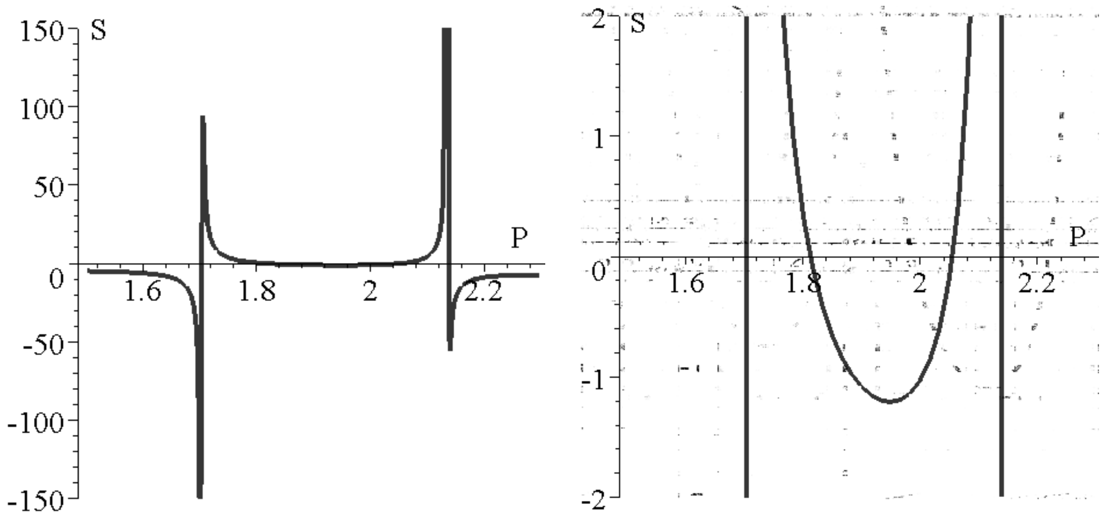


Fig. 3 Graph of functions *S(p)*.

The feature of graphic *S (p)* is that if you change elements of the family at a qualitative level in its composition have to be involved linear elements (related to jumps function *S (p)),* parallel to the vertical axis. And characteristically, on the horizontal axis coordinate these segments have values that correspond to the critical value of the control parameter *p*. In this case, *p* = 1.7 and *p* = 2.15 (Fig. 3), which coincides with the critical values of parameter, obtained after observing for animated images.

2. *The finding of critical values of the parameters of phase trajectories.* This is the main question of the definition phase curvature of trajectories, which generally are not known descriptions of analytical formulas (like formula (2)). It coordinates on the phase trajectories are calculated mainly wit help of numerical methods of solving differential equations. So the question arises determine the phase trajectory of curvature, according to the set of *N* points (x_i, y_i) when *i* = 2..*M*-1. Main parameter is more marked as the *p*.

Choose the phase curve and three adjacent points (x_{i-1}, y_{i-1}) (x_i, y_i) and (x_{i+1}, y_{i+1}). For approximate calculation of curvature at the point (x_i, y_i) we find the radius of the circle r_1, which passes through these three points. Then the value of curvature is k=1/r_1.

We use known from analytic geometry equation circles expressed using the determinant

$$\begin{vmatrix} u^2+v^2 & u & v & 1 \\ x_{i-1}^2+y_{i-1}^2 & x_{i-1} & y_{i-1} & 1 \\ x_i^2+y_i^2 & x_i & y_i & 1 \\ x_{i+1}^2+y_{i+1}^2 & x_{i+1} & y_{i+1} & 1 \end{vmatrix} = 0 . \qquad (4)$$

We denote $A = \begin{vmatrix} x_{i-1} & y_{i-1} & 1 \\ x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \end{vmatrix}$;

$$B = y_{i+1}^2 y_i - x_{i+1}^2 y_{i-1} + x_{i+1}^2 y_i - y_{i+1}^2 y_{i-1} - y_i^2 y_{i+1} + y_i^2 y_{i-1} +$$
$$+ x_i^2 y_{i-1} - x_t^2 y_{i+1} - y_{i-1}^2 y_i + y_{i-1}^2 y_{i+1} - x_{i-1}^2 y_i + x_{i-1}^2 y_{i+1};$$
$$C = y_{i+1}^2 x_{i-1} - y_{i+1}^2 x_i + x_{i+1}^2 x_{i-1} - x_{i+1}^2 x_i + y_i^2 x_{i+1} - y_i^2 x_{i-1} +$$
$$+ x_i^2 x_{i+1} - x_t^2 x_{i-1} + y_{i-1}^2 x_i - y_{i-1}^2 x_{i+1} + x_{i-1}^2 x_i - x_{i-1}^2 x_{i+1};$$
$$D = y_{i+1}^2 x_i y_{i-1} - y_{i+1}^2 x_{i-1} y_i - x_{i+1}^2 x_{i-1} y_i + x_{i+1}^2 x_i y_{i-1} +$$
$$+ y_i^2 x_{i-1} y_{i+1} - y_i^2 x_{i+1} y_{i-1} - x_i^2 x_{i+1} y_{i-1} + x_t^2 x_{i-1} y_{i+1} -$$
$$- y_{i-1}^2 x_{i+1} y_i - y_{i-1}^2 x_i y_{i+1} - x_{i-1}^2 x_i y_{i+1} + x_{i-1}^2 x_{i+1} y_i.$$

As a result, we obtain the formula to calculate the radius of the circle

$$r_i = \frac{\sqrt{\left(\frac{B}{2}\right)^2 + \left(\frac{C}{2}\right)^2 - AD}}{A}, \qquad (5)$$

and the coordinates of its center

$$x_{tЦ} = -\frac{B}{2A}; \qquad y_{iЦ} = -\frac{C}{2A}. \qquad (6)$$

Based on the formula (5) we can calculate the approximate value of the curvature $k_i = 1/r_i$ and at the point (x_i, y_i). Conditions $A = 0$ determine the zero curvature (when points are on the line).

After determining Krivin for all points on the phase trajectory we build piecewise linear graph of *k (t)* of curves for a parameter *p*. Square of graph (for the initial value of *p*) is determined by using one of the numerical methods (eg, Simpson's rule). In carrying out these actions in the loop for other values of p, we get an approximate graph of *S (p),* composed of linear segments. According to the above, the critical value determined with help of vertical components of piecewise linear graph *S (p).*

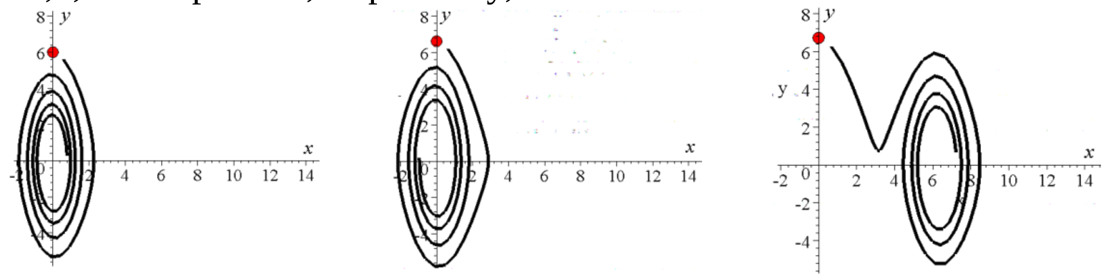
3. *Investigation of oscillations of a mathematical pendulum*, whose state is described by a system of differential equations [5]:

$$\frac{d}{dt}x(t) = y(t); \qquad \frac{d}{dt}y(t) = -0{,}2y(t) - 9.8\sin(x(t)) \qquad (7)$$

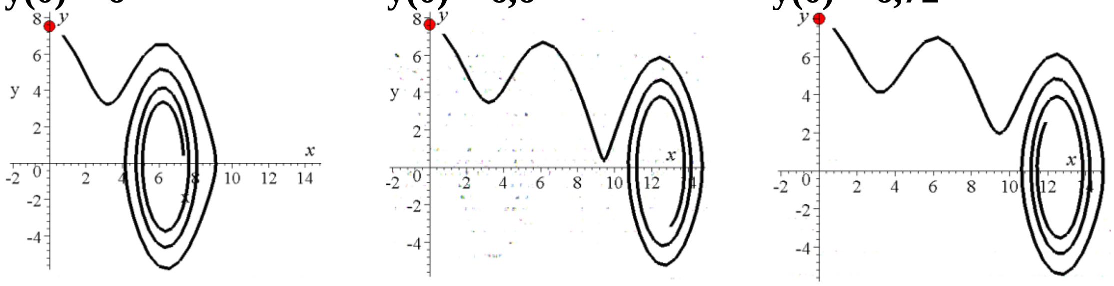
On the qualitative level show that "critical" parameter of mathematical pendulum may be the initial velocity of its motion, and is defined in the initial condition (0). Depending on the size of (0) oscillations it will either be damped or "rotating" around the suspension point. That change is the certain value of the initial velocity (0) and can distinguish between the "quality" of oscillation.

We will solve the system of differential equations (7) numerically with help of Runne-Kuta method with initial conditions *x (0)* and *y(0)*. To test the program (compiled in the language environment Maple) was selected condition *x (0) = 0* and *6 <y (0) <8.*

Sketch. 4 shows animated footage of the phase portrait changes of the critical values are determined by the initial velocity $y_1(0) = 6{,}65$ and $y_2(0) = 7{,}6$, which provide, respectively, one or two rotation.



y(0) = 6 **y(0) = 6,6** **y(0) = 6,72**



y(0) = 7,52 **y(0) = 7,64** **y(0) = 8**

Fig. 4. Animated phase of portrait shots changes depending on *y (0).*

That is, when critical values of pendulum must carry out a rotation around the point of suspension. Point is indicated by circles on the figures , which corresponds to the initial conditions. In this step between points on the phase curve was selected Δ = 0,1; the number of points *N* = 1000.

Sketch. 5 shows the graph of *Y (p)* depending on the number of iterations *N*, where certain critical values *y1 (0) = 6.65 and y2 (0) = 7.6.*

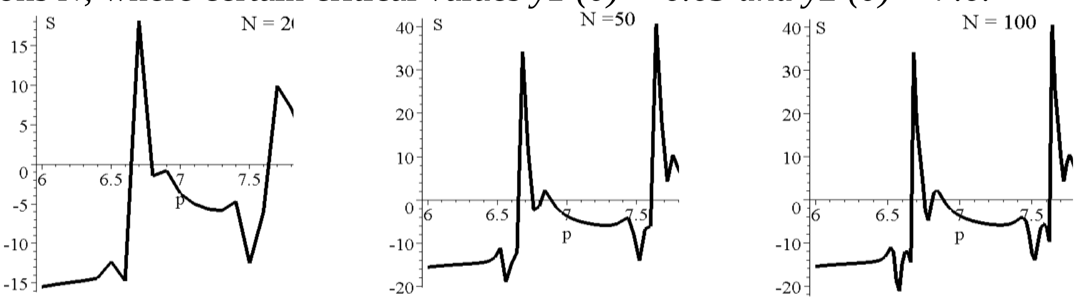
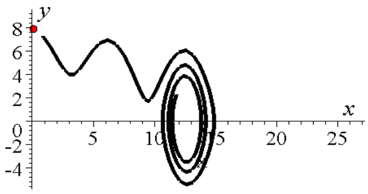


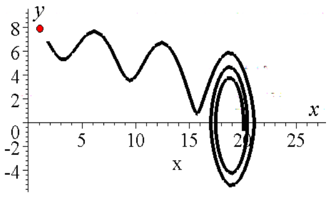
Fig. 5. Graph of function *S (p)* depending on the number of iterations *N*.

The continuing provision of critical points with values N = 50 and N = 100 indicates correct convergence process calculations.

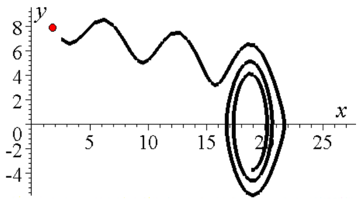
More interesting is the case of the analysis of fluctuations on conditions *0 <x (0) <6* and *y (0) = 8*. Fig. 6 shows the phase portrait shots changes depending on the values of *x (0)*. Sketch. 7 shows a graph corresponding function S (p) the critical values of $x_1(0)$ =1, $x_2(0)$ =2 i $x_3(0)$ =4,8.



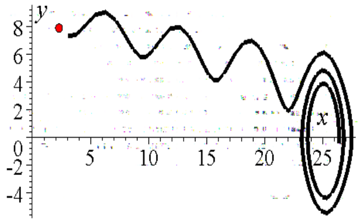
x(0) = 0.1



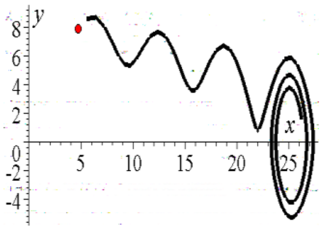
x(0) = 1,07



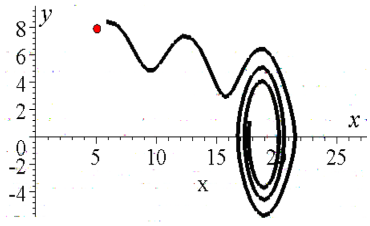
x(0) = 1,8



x(0) = 2,28



x(0) = 4,72



x(0) = 5

Fig. 6. Animated phase of portrait shots changes depending on *x (0).*

Using of the approximate calculation method using curvature of the circle drawn radius through the three points, it was necessary that the numerical method to compute approximate managed only to determine the function and its derivative. And for the curvature calculation must also know the second derivative. Therefore, after analyzing common numerical methods of integration of differential equations, it was decided to calculate the approximate radius of curvature using "tangent" circle.

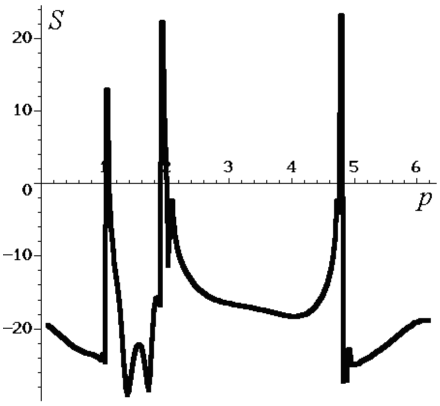


Figure 7. Chart features of *S(p)*

Conclusion. To determine the critical values of the phase trajectories of the *p* family we must find the coordinates on the axis vertical of the graph components *S = S (p)* depending on the parameter, the functions integral curvature *k (t)* (or jumps of the function *S (p))*

Literature

1. *Понтрягин Л.С.* Обыкновенные дифференциальные уравнения / Л.С.Понтрягин.- М.: Наука, 1974. – 331 с.
2. *Баутин Н.Н.* Методы и приемы качественного исследования динамических систем на плоскости / Н.Н.Баутин, Е.А Леонтович - М.: Наука, 1990. – 287с.
3. *Китаев Д.Б.* Развитие качественной теории дифференциальных уравнений в XIX столетии: дис… канд. техн. наук: 07.00.10 / Д.Б. Китаев - М: Институт истории естествознания и техники им. С.И.Вавилова РАН. – 2011. – 140 с.
4. *Мищенко А.С.* Сборник задач по дифференциальной геометрии и топологии / А.С.Мищенко, Ю.П.Соловьев , А.Т.Фоменко – М: Изд. ФМЛ, 2001. – 352 с.
5. *Бутенин Н.В.* Теория колебаний / Н.В.Бутенин - М.: Высшая школа, 1963. – 187 с.
6. *Шуп Т.* Решение инженерных задач на ЭВМ: Практическое руководство / Т.Шуп - Пер. с англ. - М.: Мир, 1982 -238с.
7. *Бидерман В.Л.* Прикладная теория механических колебаний / В.Л.Бидерман - М.: Высшая школа, 1972. – 416 с.