UDC 7.048.38

PLANE'S MOTIONS DESCRIBING THE CONSTRUCTION OF A FIGURED TILE OF AN ORNAMENT ON M. C. ESCHER'S LITHOGRAPH 'REPTILES', AND ITS SYMMETRY GROUP

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In the foreign press you can find numerous articles whose authors attempt to answer the question: 'How did M. C. Escher create such famous prints as 'Horsemen', 'Day and Night', 'Sky and Water' or 'Reptiles'? It is noteworthy that all their attempts boiled down to the fact that they placed rhombuses, squares, regular triangles or regular hexagons on the print, cut out a repetitive fragment from it with their help and completely filled the plane with it.

In our opinion, the method of regularly dividing a plane, consisting in the fact that a repetitive pattern fits into some regular polygon, not only does not lead to an explanation of how M. C. Escher worked, but also leads away from it in the opposite direction. Therefore, before looking in the prints of M. C. Escher for fragments that fit into rhombuses, squares, regular triangles, and so on, it is necessary to understand how M. C. Escher created figures that, with the help of its translations, rotations or mirror reflections cover the plane without overlaps and gaps.

Thus, our purpose is to classify ornaments according to crystallographic symmetry groups on the plane, discovered by the Russian scientist E. S. Fedorov, and to connect the symmetry groups of ornaments with the groups of plane's motions that describe the construction of their repetitive figures.

A rule has been proposed for constructing figured tiles that stylize images of plants and animals and fills the plane without overlaps and gaps during translations and rotations of its repetitions, in particular a figured tile that generalizes the zoomorphic form in M. C. Escher's lithograph 'Reptiles'. The construction of a figured tile that generalizes the zoomorphic form in M. C. Escher's lithograph 'Reptiles' is considered. The proposed rule was applied to compose an ornament stylizing M. C. Escher's lithograph 'Reptiles'. It is shown that this ornament has set of axes of 3rd order symmetry and six translation axes. A connection has been revealed between the symmetry group of the ornament and the motions of the plane leading to the formation of its figured tiles. It is assumed that our next work will be devoted to the application of one of the E. S. Fedorov's crystallographic symmetry groups to the construction of a figured tile that stylizes a zoomorphic form in one of the graphic works of M. C. Escher.

Keywords: tessellation of a plane, figured tiles in the form of animals and

plants, stylization of M. C. Escher's prints.

Formulation of the problem. The problem of covering a plane with figures of the same shape without overlaps and gaps came to us from ancient times. She is at least 5 thousand years old. Ornaments consisting of stylized figures of animals and plants that completely fill the plane were widespread in the decorative art of the Ancient East, in particular in the art of Ancient Egypt and Persia. However, they did not become the subject of study by either ancient Egyptian or Persian mathematicians. European scientists paid attention to them after an exhibition of graphics by the outstanding Dutch artist Maurits Cornelis Escher (1898–1972) was held in Rotterdam in 1949, where viewers saw for the first time how the figures of birds, fish, reptiles and other animals fit into each other without any overlaps and gaps. He was inspired to create them by his knowledge in 1936 of the stucco ornaments and tiles of the Alhambra, the palace of the Muslim rulers of Spain from the Nasrid dynasty (1230–1492), towering over the eastern outskirts of Granada, the former capital of the Emirate of Granada [1]. By the way, we believe that the Arabs were the same discoverers of the 'Moorish style' as they were the discoverers of the 'Arabic numerals'. In our opinion, the interior chambers of the Alhambra were decorated by Persian craftsmen brought by the Arabs to Spain. The Russian writer V. P. Botkin, who visited the Alhambra in 1845, writes about this impression in his travel notes 'Letters about Spain': 'The walls of the hall are covered with painted arabesques... At first glance, it seems as though the ceiling and walls are covered with Persian carpets or embroidered canvas wallpaper with the smallest patterns'.

Analysis of recent research and publications. After the work of M. C. Escher entered scientific use, not a single monograph devoted to symmetry in nature, in particular in crystals, was published without illustrations reproducing his prints [2–13]. Scientists have discovered that in the mosaics of M. C. Escher, formed by repetitions of the same figure, all 17 symmetry groups on a plane are found, strictly mathematically derived by the outstanding Russian crystallographer E. S. Fedorov and published by him in 1891 in the monograph 'Symmetry on plane'. Additionally, a passion for the work of M. C. Escher led the outstanding English mathematician Roger Penrose to the discovery in 1974 of three types of non-periodic, that is, non-translationally symmetrical tiling of the plane with irregular polygons, called 'Penrose tiles' in his honor. Finally, numerous articles began to appear in the foreign press, the authors of which attempted to answer the question: 'How did M. C. Escher create such famous prints as 'Horsemen', 'Day and Night', 'Sky and Water' or 'Reptiles'? It is noteworthy that all their attempts boiled down to the fact that they placed rhombuses, squares, regular triangles or regular hexagons on the print, cut out a repetitive fragment from it with their help and completely filled the plane with it [2–8]. Moreover, one of the authors even stated that he was able to generalize the way in which M. C. Escher created his prints [8].

In our opinion, the method of regularly dividing a plane, consisting in the fact that a repetitive pattern fits into some regular polygon, not only does not lead to an explanation of how M. C. Escher worked, but also leads away from it in the opposite direction. Firstly, when M. C. Escher gave explanations to his prints to visitors to the famous exhibition in 1949, he admitted that in order to select the shape of a figure, for example a reptiles, he sculpted it from plasticine, moved it along the plane of a sheet of paper and from time to time corrected its shape [1]. Consequently, the creation of a figure that completely fills the plane is the initial stage of the work, and the construction of an ornament with its help is the final stage of the work. Secondly, the method of regularly division of the plane proposed by foreign authors does not answer the main question: ‘How to draw an image contained in a regular polygon and covering the plane without overlaps and gaps by translations of its repetitions’? Thirdly, the method of regularly division of a plane ‘discovered’ by foreign authors has long been known, and in decorative painting of fabrics there is a regular polygon into which the design fits in such a way that its part adjacent to one side of the polygon is a continuation of its part adjacent to the opposite side of the same polygon is called rapport [14].

Therefore, before looking in the prints of M. C. Escher for fragments that fit into rhombuses, squares, regular triangles, and so on, it is necessary to understand how M. C. Escher created figures that, with the help of its translations, rotations or mirror reflections cover the plane without overlaps and gaps. Unfortunately, at present we do not even see attempts to unravel the mystery of the construction of repetitive figures in the prints of M. C. Escher. Meanwhile, if we knew what transformations needed to be performed in order to construct a figure that completely covers the plane, then it would be possible to establish a connection between the group of transformations related to the figure with the group of transformations related to the ornament obtained by translations, rotations and mirror reflections of it repetitions. Therefore, the definition of groups of plane movements related to the repetitive figures of ornaments created by M. C. Escher and their classification is an actual challenge in the theory of ornament.

Formulating the purposes of the article. Thus, our purpose is to classify ornaments according to crystallographic symmetry groups on the plane, discovered by the Russian scientist E. S. Fedorov, and to connect the symmetry groups of ornaments with the groups of plane’s motions that describe the construction of their repetitive figures.

Main part. Let us consider the construction of a figure that stylizes a zoomorphic form in M. C. Escher’s lithograph ‘Reptiles’ and fills the plane without overlaps and gaps with translations and rotations of its repetitions [15–17].

Let the reader not be confused by the fact that in the lithograph ‘Reptiles’ regular hexagons are inscribed in zoomorphic forms. Apparently, M. C. Escher wanted to create the impression that if a regular hexagon contains a pattern applied according to certain rules, then by moving it along the plane of the paper, one can create an ornament, parts of which are periodically repeated in one or more directions. Of course, regular hexagons with a zoomorphic shape inscribed in them really fill the plane without overlaps and gaps and form the ornament presented in the lithograph ‘Reptiles’. However, the problem is that the regular hexagons were applied to the ornament by M. C. Escher after the ornament was created. Consequently, in order to build rapport in the form of a regular hexagon with a zoomorphic form inscribed in it, it was necessary to first create a zoomorphic form, and then only inscribe it into the regular hexagon.

How to obtain the zoomorphic form that is contained in the regular hexagons that form the ornament represented in M. C. Escher’s lithograph ‘Reptiles’? We have found a surprisingly simple rule for constructing a figure that stylizes the zoomorphic form in M. C. Escher’s lithograph ‘Reptiles’ and fills the plane without overlaps and gaps with translations and rotations of its repetitions. However, due to the limited space of the article, we cannot describe it in it.

Let us apply the rule we have found to the composition of an ornament stylizing M. C. Escher’s lithograph ‘Reptiles’ and show it in Fig. 1, a its first variant.

It is remarkable that the ornament shown in Fig. 1, a, is a figure that fits into a regular hexagon. Therefore, it can be considered as the result of tiling the plane with groups of zoomorphic forms, carried out by translations in six directions, defined by perpendiculars to the sides of a regular hexagon.

Let’s consider the types of symmetry that parquet, built on the basis of a group of zoomorphic forms according to the first variant, has.

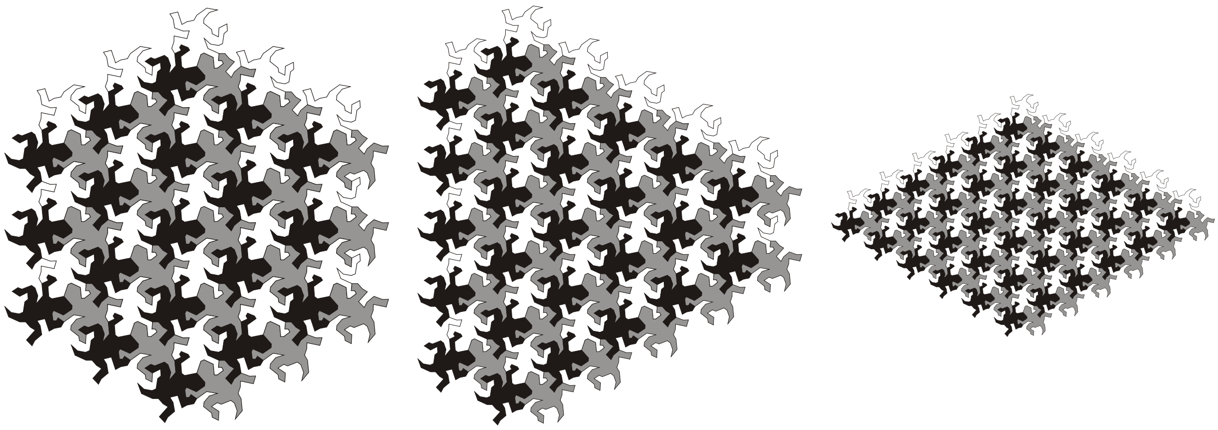
It is obvious that the ornament shown in Fig. 1, a does not have a single plane of symmetry, that is, it does not have a single reflection symmetry group. At the same time, it has rotational symmetry with 3rd order symmetry axes. This means that the ornament we are considering can be superposed with itself by rotating it around the axis of symmetry at an angle of 120°. Additionally, the ornament shown in Fig. 1, a, has translation symmetry with six translation axes defined by perpendiculars to the sides of a regular hexagon. Consequently, it can be superposed with itself using translation in one of the six directions defined by the translation axes.

Let’s present the ornament shown in Fig. 1, a, as a result of tiling the plane with groups of zoomorphic forms, carried out by translations in three directions, defined by perpendiculars to the sides of a regular triangle. Let's show in Fig. 1, b is the second version of the ornament, stylizing M. C. Escher’s lithograph ‘Reptiles’ and representing a figure that fits into a regular triangle.

Let’s consider the types of symmetry that parquet, built on the basis of a group of zoomorphic forms according to the second variant, has.

It is obvious that the ornament shown in Fig. 1, b does not have a single plane of symmetry, that is, it does not have a single reflection symmetry group. At the same time, it has rotational symmetry with 3rd order symmetry axes. This means that the ornament we are considering can be superposed with itself by rotating it around the axis of symmetry at an angle of 120°. Additionally, the ornament shown in Fig. 1, b, has translation symmetry with three translation axes defined by perpendiculars to the sides of a regular triangle. Consequently, it can be superposed with itself using translation in one of the three directions defined by the translation axes.

Let us now present the ornament shown in Fig. 1, a, as a result of tiling the plane with groups of zoomorphic forms, carried out by translations in four directions, defined by perpendiculars to the sides of the rhombus. Let’s show in Fig. 1, c the third variant of the ornament, stylizing the lithograph ‘Reptiles’ and representing a figure that fits into a rhombus.



a b c

Fig. 1. Ornament, stylizing of M. C. Escher’s lithograph ‘Reptiles’

Let’s consider the types of symmetry that parquet, built on the basis of a group of zoomorphic forms according to the third variant, has.

It is obvious that the ornament shown in Fig. 1, c does not have a single plane of symmetry, that is, it does not have a single reflection symmetry group. At the same time, it does not have a single axis of symmetry, that is, it does not have a single rotation symmetry group. This means that the ornament we are considering can be superposed with itself by rotating around the axis of symmetry only by an angle of 360°. At the same time, the ornament shown in Fig. 1, c, has translation symmetry with four translation axes defined by perpendiculars to the sides of the rhombus. Consequently, it can be superposed with itself with translation in one of the four directions defined by the translation axes.

It is remarkable that, depending on the method of laying the same figured tile, we get three types of parquet, which differ from each other in that, firstly, they fit into different regular polygons, namely: regular hexagon, regular triangle and rhombus, - and, secondly, they are described by different symmetry groups.

Shown in Fig. 1, a, the ornament clearly proves that the rule we found for constructing figured tiles in the image of a group of zoomorphic forms can be considered a law to which all ornaments that satisfy the following conditions are subjects:

zoomorphic shapes form a group having an axis of symmetry of the 3rd order;

the centers of rotation of groups of zoomorphic forms are located at the vertices of regular hexagons inscribed in circles, the diameters of which form an arithmetic progression with a difference equal to three times the diameter of the original circle;

groups of zoomorphic shapes can be combined with each other if translations are carried out in directions defined by straight lines passing through the vertices of regular hexagons, over distances equal to three times the radius of the original circle.

Let us pay attention to the connection that exists between the ornament stylizing M. C. Escher’s lithograph ‘Reptiles’ and his figured tiles. The connection is that both the symmetry of the ornament and its figured tiles are described by groups of rotation of the 3rd order and groups of translations of the axes of rotation. It follows that if any figure corresponds to any group of plane transformations, then the same group of plane transformations will correspond to an ornament obtained by the same figure transformations on the plane. Indeed, a zoomorphic shape is formed by rotations around axes of the 3rd order, and the axes of rotation are subject to translations, transferring them from the center of a regular hexagon to one of its vertices. Meanwhile, the ornament is formed by rotations of groups of zoomorphic forms around axes of the 3rd order, and the axes of rotation are subject to translations, transferring them from one vertex of a regular hexagon to the vertex of another regular hexagon. In our opinion, this proves that there is an indissoluble connection between the symmetry group of the ornament and the movements of the plane leading to the formation of its figured tiles.

Conclusions. Thus, the article proposes a rule for constructing a figured tile that stylizes images of plants and animals and fills the plane without overlaps and gaps during translations and rotations of its repetitions, in particular a figured tile that generalizes the zoomorphic form in M. C. Escher’s lithograph ‘Reptiles’. The proposed rule was applied to compose an ornament stylizing M. C. Escher’s lithograph ‘Reptiles’. It is shown that this ornament has set of axes of 3rd order symmetry and six translation axes. A connection has been revealed between the symmetry group of the ornament and the motions of the plane leading to the formation of its figured tiles. It is assumed that our next work will be devoted to the application of one of the E. S. Fedorov’s crystallographic symmetry groups to the construction of a figured tile that stylizes a zoomorphic form in one of the graphic works of M. C. Escher.

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РУХИ ПЛОЩИНИ, ЩО ОПИСУЮТЬ ПОБУДУВАННЯ ФІГУРНОЇ ПЛИТКИ ОРНАМЕНТУ НА ЛІТОГРАФІЇ М. К. ЕШЕРА «ЯЩІРКИ», ТА ЙОГО ГРУПА СИМЕТРІЇ

Ніцин О.Ю.

У зарубіжній пресі можна знайти численні статті, автори яких роблять спроби відповісти на запитання: «Як М. К. Ешер створював такі знамениті гравюри, як «Вершники», «День і ніч», «Небо та вода» чи «Ящірки»?» Примітно, що всі їхні спроби зводяться до того, що на гравюру вони накладають ромби, квадрати, правильні трикутники або правильні шестикутники, вирізують з їх допомогою фрагмент, що повторюється, і заповнюють їм всю площину.

На нашу думку, спосіб регулярного розбиття площини, що полягає в тому, що малюнок, що повторюється, вписується в який-небудь правильний багатокутник, не тільки не призводить до пояснення того, як працював М. К. Ешер, а й відводить від нього в протилежний бік. Тому перш ніж шукати в гравюрах М. К. Ешера фрагменти, які вписуються в ромби, квадрати, правильні трикутники і так далі, необхідно зрозуміти, яким чином М. К. Ешер створював постаті, які за допомогою її паралельних переносів, обертань чи дзеркальних відбитків покривають площину без накладень та перепусток.

Таким чином, наша мета полягає в тому, щоб класифікувати орнаменти по кристалографічним групам симетрії на площині, відкритим російським ученим Є. С. Федоровим, і зв'язати групи симетрії орнаментів з групами рухів площини, що описують побудову їх фігур, що повторюються.

Запропоновано правило побудови фігурної плитки, що стилізує зображення рослин і тварин та заповнює площину без накладень та перепусток при паралельних переносах та обертаннях її повторень. Розглянуто побудову фігурної плитки, що узагальнює зооморфну форму на літографії М. К. Ешера «Ящірки». Запропоноване правило було застосовано для складання орнаменту, що стилізує літографію М. К. Ешера «Ящірки». Показано, що цей орнамент має безліч осей симетрії 3-го порядку та шість осей перенесення. Припущено, що предметом подальших досліджень буде додаток однієї із кристалографічних груп симетрії Є. С. Федорова до побудови фігурної плитки, що стилізує зооморфну форму на одній із графічних робіт М. К. Ешера.

Ключові слова: замощення площини, фігурні плитки у формі тварин і рослин, стилізація гравюр М. К. Ешера.

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