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MODELING OF PH-CURVES IN THE FORM OF THE BASIC SPLINE

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The work elucidates the way of constructing Pythagorean-Hodograph curves (PH-curves) in the form of the basic spline. The conditions for determining points of the cubic spline curves are found. The examples of curves are given.

Keywords: basic spline, Catmull–Rom spline, PH-curve, Pythagorean-Hodograph Curves.

Formulation of the problem. To designate a curve with a specific arc length, it is expedient to use a class of curves for the Pythagorean hodograph (PH-curves) [1, 2]. The arc length of such curves can be calculated without numerical methods, the formula for the calculation has the form of a polynomial. In many systems of computer graphics for the construction of curves and surfaces, the information of the point series is used, so the urgent question is the study of the representation of curves for the Pythagorean hodograph in the form of a fundamental spline [3].

Analysis of recent research and publications. Professor Reed Farrouki and his co-authors [1, 2] investigate the use of curves for the Pythagorean hygrogram (PH) in geometric design, graphing, planning and driving. The theory, algorithms and the use of plane and spatial PH-curves are considered. The work [4] seeks to study and construct an isotropic curve for the Pythagorean hodograph in the form of a Bezier curve of the third order. In the following [5], this curve is used to simulate a planar orthogonal and isothermal grid. The author of the paper [6] studies the construction of an isotropic spatial fundamental spline, and finds the conditions of isotropy.

Formulating the goals of the article. The purpose of this work is to develop a method for constructing an RN curve in the form of a fundamental spline.

Main part. Flat curve $r(u)=[x(u) \quad y(u)]$ will be a curve for the Pythagorean hodograph (PH-curve) if and only if the hodograph (derivatives) from $r(u)$ linked by the following ratio [2]:

$$|r'(u)|^2 = x'(u)^2 + y'(u)^2 = \sigma(u)^2 \tag{1}$$

for some polynomial $\sigma(u)$.

For expression (1), conditions were found for which is polynomial the curve may be a PH curve:

$$x'(u) = w(u)(f(u)^2 - v(u)^2),$$
$$y'(u) = 2w(u)f(u)v(u), \qquad (2)$$

where $f(u)$, $v(u)$, $w(u)$ - polynomials.

If we substitute the value (2) in (1) provided $w(u)=1$, then we will have:

$$x'(u)^2 + y'(u)^2 = (f(u)^2 + v(u)^2)^2. \qquad (3)$$

We find dependences for a plane fundamental spline such that the condition (3) is fulfilled, that is, the curve turned into a curve for the Pythagorean hodograph, but was determined based on the values of the point frame.

We will look for a cubic curve $f(u)$ and $v(u)$ in the form of linear polynomials:

$$f(u) = a_0 + a_1 u, \qquad (4)$$
$$v(u) = b_0 + b_1 u,$$

where a_0, a_1, b_0, b_1 - some numbers.

The plot of the fundamental spline is given by the position of the four points of the given point frame, and the tangents at each point are calculated by the coordinates of two adjacent points. Let the fundamental spline be given in the form [3]:

$$P(u) = [(P_{k+1} - P_{k-1})(u^3 - 2u^2 + u) + (P_{k+2} - P_k)(u^3 - u^2)]s + \qquad (5)$$
$$+[P_k(2u^3 - 3u^2 + 1) + P_{k+1}(-2u^3 + 3u^2)],$$

where P_{k-1}, P_k, P_{k+1}, P_{k+2} - points of a given point frame (fig.1),

$s = \frac{1-t}{2}$, t- tension parameter spline.If $t = 0$, We have a kind of fundamental spline, namely, the splintCatmall-Roma (Catmull-Romslines).

Take the derivative of the expression (5):

$$P'(u) = (P_{k+1} - P_{k-1})s + 2u[-2s(P_{k+1} - P_{k-1}) - s(P_{k+2} - P_k) + 3(P_{k+1} - \qquad (6)$$
$$- P_k)] + 3u^2[s(P_{k+1} - P_{k-1}) + s(P_{k+2} - P_k) - 2(P_{k+1} - P_k)].$$

Let's substitute meaning $f(u)$, $v(u)$ from the equation (4) in the condition for the PH-curve (2). We'll get it :

$$x'(u) = a_0{}^2 - b_0{}^2 + 2u(a_0 a_1 - b_0 b_1) + u^2(a_1{}^2 - b_1{}^2), \qquad (7)$$
$$y'(u) = 2a_0 b_0 + 2u(a_0 b_1 - a_1 b_0) + 2u^2 a_1 b_1.$$

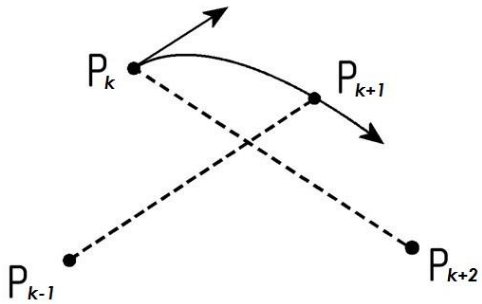

Fig. 1. Formation of a fundamental spline

We will determine the coordinates of the point $P_{k-1}(x_{k-1}, y_{k-1})$. To find the coordinates of the points $P_k(x_k, y_k), P_{k+1}(x_{k+1}, y_{k+1}), P_{k+2}(x_{k+2}, y_{k+2})$ equate the coefficients for the corresponding coefficients from equations (6) and (7). We place the system of equations:

$$\begin{aligned}
&a_0^2 - b_0^2 = s(x_{k+1} - x_{k-1}),\\
&a_0a_1 - b_0b_1 = 2sx_{k-1} + (s-3)x_k + (3-2s)x_{k+1} - sx_{k+2},\\
&a_1^2 - b_1^2 = 3(sx_{k+2} + (s-2)x_{k+1} + (2-s)x_k - sx_{k-1}),\\
&2a_0b_0 = s(y_{k+1} - y_{k-1}),\\
&a_0b_1 + a_1b_0 = 2sy_{k-1} + (s-3)y_k + (3-2s)y_{k+1} - sy_{k+2},\\
&2a_1b_1 = 3(sy_{k+2} + (s-2)y_{k+1} + (2-s)y_k - sy_{k-1}).
\end{aligned} \tag{8}$$

Solving the system of equations (8) we obtain the coordinates of the points of the given point frame:

$$\begin{aligned}
&x_{k+1} = \frac{a_0^2 - b_0^2}{s} + x_{k-1},\\
&y_{k+1} = \frac{2a_0b_0}{s} + y_{k-1},\\
&x_k = x_{k+1}(1-s) + sx_{k-1} - a_0a_1 + b_0b_1 - \frac{a_1^2 - b_1^2}{3},\\
&y_k = y_{k+1}(1-s) + sy_{k-1} - a_0b_1 - a_1b_0 - \frac{2a_1b_1}{3},\\
&x_{k+2} = \frac{a_1^2 - b_1^2}{3s} - (1 - \frac{2}{s})x_{k+1} - (\frac{2}{s} - 1)x_k + x_{k-1},\\
&y_{k+2} = \frac{2a_1b_1}{3s} - (1 - \frac{2}{s})y_{k+1} - (\frac{2}{s} - 1)y_k + y_{k-1}.
\end{aligned} \tag{9}$$

Example 1. We build the Katmall-Rome splint on the 3rd order with the Pythagorean hodograph (fig. 2). We will ask: $f(u) = 2 + 3u, v(u) = -2 - u,\ P_{k-1} = (1,1)$.

On the basis of the obtained equations (9) find the coordinates of all points for a given point frame:

$P_k(x_k, y_k) = (-3{,}3)$; $P_{k+1}(x_{k+1}, y_{k+1}) = (1{,}-15)$;
$P_{k+2}(x_{k+2}, y_{k+2}) = (13{,}-57)$.

Now we find the length of the splint of Cattmall-Rome 3rd order by the Pythagorean hodograph. To do this, skipigayutsya expression (3):

$$L = \int_0^1 \sqrt{x'(u)^2 + y'(u)^2}\, du = \int_0^1 \sqrt{(f(u)^2 + v(u)^2)^2}\, du = \int_0^1 (f(u)^2 + v(t)^2)\, du. \quad (10)$$

As can be seen from condition (10), this representation of the curve allows you to get rid of the root, that is, to squeeze the exact value of the integral. For a fundamental spline of the third order, we must have it:

$$\begin{aligned} L = {} & su(x_{k+1} - x_{k-1}) + u^2(2sx_{k-1} + (s-3)x_k + (3-2s)x_{k+1} - sx_{k+2}) \\ & + u^3(sx_{k+2} + (s-2)x_{k+1} + (2-s)x_k - sx_{k-1}). \end{aligned} \quad (11)$$

Calculate the length of the split Cattmall-Rom for Example 1. To do this, we substitute the expression (11) for the value. We will have: $L = 4$.

Example 2. We build the splitCatmall-Rome 3rd order by the Pythagorean hodograph on the basis of given points of the frame $P_{k-1} = (1{,}1)$, $P_k = (-7{,}10/3)$, $P_{k+1} = (-5{,}9)$ (fig. 3).

On the basis of equations (8) and numerical methods, we find coefficients a_0, a_1, b_0, b_1 for linear polynomials: $b_0 = 2$; $a_0 = 1$; $b_1 = -1$; $a_1 = 2$.

We substitute the resulting coefficients in (9) to find the coordinates of the last point of the frame. We'll get it

$P_{k+2}(x_{k+2}, y_{k+2}) = (9{,}15.3333)$. Spline length $L = 2$.

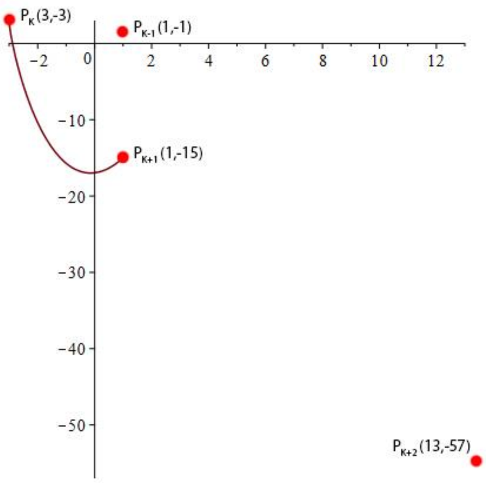

Fig. 2. Spline for the hodograph Pythagorean based on given linear functions

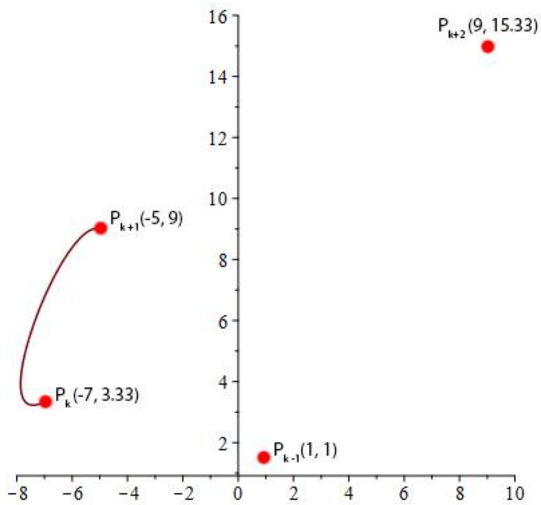

Fig. 3. Splint for a hodograph Pythagoras based on given points

Conclusions. As a result of the performed research, it was proposed to construct a curve for the Pythagorean hodograph in the form of a fundamental spline. The length of such a curve can be determined without the use of approximation methods. Two approaches to the formation of a fundamental spline were considered: the first one - through the problem of coefficients of linear functions, the second - through the problem of spline points.

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МОДЕЛИРОВАНИЕ PH-КРИВЫХ НА ОСНОВЕ ФУНДАМЕНТАЛЬНОГО СПЛАЙНА

Аушева Н.Н, Мельник О.В., Гомов В.В.

В работе рассматривается способ построения пространственных кривых по годографу Пифагора (PH-кривые) на основе фундаментального сплайна. Найдены условия для определения точек фундаментального сплайна. Приведены примеры кривых.

Ключевые слова: фундаментальный сплайн, сплайн Катмалл-Рома, PH-кривая, кривая по годографу Пифагора.

МОДЕЛЮВАННЯ PH-КРИВИХ У ВИГЛЯДІ ФУНДАМЕНТАЛЬНОГО СПЛАЙНУ

Аушева Н.М., Мельник О.В., Гомов В.В.

Робота висвітлює спосіб побудови просторових кривих за годографом Піфагора (PH-кривих) у вигляді фундаментального сплайну. Знайдено умови для визначення точок сплайну для кубічних кривих. Наведено приклади кривих.

Ключові слова: фундаментальний сплайн, сплайн Катмалл-Рома, PH-крива, крива за годографом Піфагора.