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BACK-SYMMETRIC MATRICES SIMILARITY RELATIVE TO SECONDARY DIAGONAL

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Back-symmetric matrices obtained as a result of paired comparisons of alternatives are analyzed in the paper. An evidence of matrices' similarity with elements scrambling relative to secondary diagonal is proposed.

Key words: back-symmetric matrices, matrices similarity, paired comparison of alternatives, analytic hierarchy process, decision making.

Formulation of the problem. In the hierarchy analysis method (MAI) [1] and a number of other methods, one of the stages of obtaining estimates is a pair comparison of alternatives. The matrices obtained as a result of this comparison are backward-symmetric. Assess the consistency of the judgments of the expert or the decision-maker (decision maker), which can be violated by human factors (incompetence, evaluation capabilities, lack of reliable information about alternatives, etc.), and drawbacks of the comparison scale used. Evaluation of consistency (OS) of expert judgments is carried out by comparing the consistency index (IP) with the random consistency index (ASC):

$$ОС = ИС / ИСС; \qquad (1)$$

$$ИС = \frac{\lambda_{\max} - N}{N - 1}, \qquad (2)$$

где $\lambda_{\max}$ – maximal eigenvalue of the matrix, and N the dimension of the matrix. ICS, in turn, is also determined based on (2) for random sampling of inversely symmetric matrices whose elements are the corresponding elements of the selected comparison scale. In different sources [2], the data for ASC differ, in view of the differences in the calculation methods, sample sizes and the investigated scale.

Analysis of recent research and publications. In [2], the results of the obtained estimates of the ASC by different authors are presented. For matrices of small dimension (N = 3-5), it was suggested in [3] to use instead of randomly selecting matrices-all possible matrices for the chosen comparison scale. The similarity of the inverse-symmetric matrices relative to the side diagonal makes it possible to reduce the number of necessary computations by 47-50%.

Formulation of the purpose of the article. In [3], the property of symmetry of inversely symmetric matrices with respect to an auxiliary diagonal was used, but it was given without due mathematical proof. In this paper we consider the proof of the similarity of these matrices and, accordingly, the correctness of the estimates obtained [3].

Main part. Consider two matrices A and B (N = 3), which differ by permuting the elements $a_{12} \to a_{23}$ = $b_{23} \to b_{12}$. Because The matrices A and B are inversely symmetric, then the elements $a_{21} = 1/a_{12}$, $a_{32} = 1/a_{23}$, $b_{21} = 1/b_{12}$, $b_{32} = 1/b_{23}$.

$$A = \begin{vmatrix} 1 & a_1/b_1 & a_2/b_2 \\ b_1/a_1 & 1 & a_3/b_3 \\ b_2/a_2 & b_3/a_3 & 1 \end{vmatrix}; \qquad B = \begin{vmatrix} 1 & a_3/b_3 & a_2/b_2 \\ b_3/a_3 & 1 & a_1/b_1 \\ b_2/a_2 & b_1/a_1 & 1 \end{vmatrix}. \quad (1)$$

The matrix A is similar to the matrix B (A ~ B) if there exists a nonsingular matrix S such that:

$$S^{-1}AS = B. \quad (2)$$

On the other hand,

$$T^{-1}BT = A. \quad (3)$$

We use the property of similar matrices: if the matrix A is similar to the matrix B (A ~ B), and the matrix B, in turn, is similar to the matrix C (B ~ C), then the matrix A is similar to C (A ~ C). To find the matrices S, S-1, T, T-1 (2.3), we reduce the matrices A and B (1) to the form Frobenius FA and FB (4) by means of similarity transformations [4].

$$\begin{gathered} F_A = M_{n-2}^{-1} M_{n-1}^{-1} A M_{n-1} M_{n-2}, \\ F_B = L_{n-2}^{-1} L_{n-1}^{-1} B L_{n-1} L_{n-2}, \end{gathered} \quad (4)$$

wherer L_{n-1}, L_{n-2}, M_{n-1}, M_{n-2} – Matrices obtained in the process of transforming the matrices A and B to the Frobenius form. In this case, if the matrix is A ~ B, then FA ~ FB and FA = FB, since Similar matrices have the same characteristic polynomials, and (2) with the account of (4) is transformed to the form (5):

$$L_{n-1} L_{n-2} M_{n-2}^{-1} M_{n-1}^{-1} A M_{n-1} M_{n-2} L_{n-2}^{-1} L_{n-1}^{-1} = B, \quad (5)$$

where $S = M_{n-1} M_{n-2} L_{n-2}^{-1} L_{n-1}^{-1}$, $S^{-1} = L_{n-1} L_{n-2} M_{n-2}^{-1} M_{n-1}^{-1}$.

Step 1. Reduction of the n-th row of the matrix A to the Frobenius form:

$$M_{n-1} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{a_3 b_2}{a_2 b_3} & \frac{a_3}{b_3} & -\frac{a_3}{b_3} \\ 0 & 0 & 1 \end{vmatrix}; M_{n-1}^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{b_2}{a_2} & \frac{b_3}{a_3} & 1 \\ 0 & 0 & 1 \end{vmatrix} \quad ;$$

$$A_{(n-1)} = M_{n-1}^{-1} A M_{n-1} = \begin{vmatrix} 1 - \frac{a_1 a_3 b_2}{a_2 b_1 b_3} & \frac{a_1 a_3}{b_1 b_3} & \frac{a_2}{b_2} - \frac{a_1 a_3}{b_1 b_3} \\ \frac{b_1 b_3}{a_1 a_3} - \frac{a_1 a_3 b_2^2}{a_2^2 b_1 b_3} & 2 + \frac{a_1 a_3 b_2}{a_2 b_1 b_3} & 1 - \frac{a_1 a_3 b_2}{a_2 b_1 b_3} \\ 0 & 1 & 0 \end{vmatrix} .$$

Step 2. Reduction of the n-1 row of the matrix A to the Frobenius form:

$$M_{n-2} = \begin{vmatrix} \frac{a_1 a_2^2 a_3 b_1 b_3}{a_2^2 b_1^2 b_3^2 - a_1^2 a_3^2 b_2^2} & \frac{-a_1 a_2 a_3 (a_1 a_3 b_2 + 2 a_2 b_1 b_3)}{a_2^2 b_1^2 b_3^2 - a_1^2 a_3^2 b_2^2} & \frac{-a_1 a_2 a_3}{a_1 a_3 b_2 + a_2 b_1 b_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} ,$$

$$M_{n-2}^{-1} = \begin{vmatrix} \frac{b_1 b_3}{a_1 a_3} - \frac{a_1 a_3 b_2^2}{a_2^2 b_1 b_3} & 2 + \frac{a_1 a_3 b_2}{a_2 b_1 b_3} & 1 - \frac{a_1 a_3 b_2}{a_2 b_1 b_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} ,$$

$$F_A = A_{(n-2)} = M_{n-2}^{-1} A_{(n-1)} M_{n-2} = \begin{vmatrix} 3 & 0 & \frac{(a_2 b_1 b_3 - a_1 a_3 b_2)^2}{a_1 a_2 a_3 b_1 b_2 b_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} .$$

Step 3. Reduction of the n-th row of the matrix B to the Frobenius form:

$$L_{n-1} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{a_1 b_2}{a_2 b_1} & \frac{a_1}{b_1} & -\frac{a_1}{b_1} \\ 0 & 0 & 1 \end{vmatrix} ; L_{n-1}^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{b_2}{a_2} & \frac{b_1}{a_1} & 1 \\ 0 & 0 & 1 \end{vmatrix} ;$$

$$B_{(n-1)} = L_{n-1}^{-1} B L_{n-1} = A_{(n-1)} .$$

Step 4. Reduction of the n-2 row of the matrix B to the Frobenius form:

$$L_{n-2} = M_{n-2}; \; L_{n-2}^{-1} = M_{n-2}^{-1};$$

$$F_B = B_{(n-2)} = L_{n-2}^{-1} B_{(n-1)} L_{n-2} = F_A = \begin{vmatrix} 3 & 0 & \dfrac{(a_2 b_1 b_3 - a_1 a_3 b_2)^2}{a_1 a_2 a_3 b_1 b_2 b_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}.$$

Step 5. Calculation of the matrix S:

$$S = M_{n-1} M_{n-2} L_{n-2}^{-1} L_{n-1}^{-1} = M_{n-1} L_{n-2} L_{n-2}^{-1} L_{n-1}^{-1} = M_{n-1} L_{n-1}^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \dfrac{a_3 b_1}{a_1 b_3} & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

For matrices of higher dimension N = 4 and N = 5, this algorithm also makes it possible to construct a nonsingular matrix S, which, however, has a much more complicated form.

Thus, the symmetry of the blocks (1 - 2) (fig. 1) with respect to an auxiliary diagonal and the similarity of the corresponding matrices for inversely symmetric matrices.

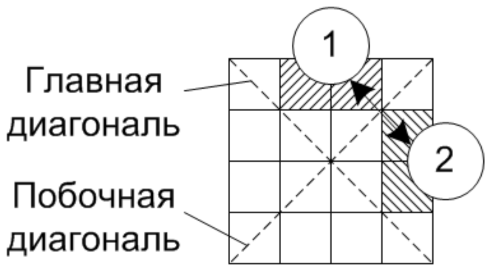

Fig. 1. Block symmetry scheme

We note that the similarity of matrices is realized only in the case of symmetry of the whole block, and not of individual elements in the blocks. Also, the elements of blocks 1-2 uniquely determine all the elements of the matrix with the exception of diagonal elements.

Conclusions. The proof of the similarity of the inverse-symmetric matrices with respect to the side diagonal allows us to significantly (up to 50%) reduce the number of matrices that need to be calculated to obtain an accurate (or with a specified accuracy) value of the random consistency index.

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