

DETERMINING THE GRADIENT OF THE SCALAR FIELD, DEFINED ON AN PLANAR DISORDERED SET OF POINTS

V. Chernyak

We consider the method of determining the gradient of the discrete scalar field defined on disordered sets of points.

Keywords: gradient, discrete scalar field, triangulation.

Formulation of the problem. Consider the conditions and method for determining the gradient of the scalar field by means of MATLAB.

Let analytically have given scalar field $u = -x^2 - y^2 + 2$. It is necessary to define the gradient field at the point (2,3).

The gradient in the classical field theory at this point is given by [1]

$$\mathit{gradu} = \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} = -2x\mathbf{i} - 2y\mathbf{j} = -2 \cdot 2\mathbf{i} - 2 \cdot 3\mathbf{j} = -4\mathbf{i} - 6\mathbf{j} \quad (1)$$

The condition means the gradient definition MATLAB - vector field must be set to discrete monotonous rectangular grid (ie unambiguous monotonically ascending or descending), and can be non-uniform grid.

By means of MATLAB coordinates of the gradient field at the top of the grid U5 (Fig. 1, a) determined by formulas

$$\frac{\partial u}{\partial x} = \frac{U_6 - U_4}{X_2 - X_1}, \quad \frac{\partial u}{\partial y} = \frac{U_2 - U_8}{Y_2 - Y_1} \quad (2)$$

In the case of a uniform grid increments $hX = hY = 1$, for example, the coordinates of the unknown point $X_0 = 2$, $Y_0 = 3$, the coordinates of the adjacent mesh nodes $X_1 = 1$, $X_2 = 3$, $Y_1 = 2$, $Y_2 = 4$, the value of the field in the grid

$$\begin{aligned} U_4 &= -X_1^2 - Y_0^2 + 2 = -1^2 - 3^2 + 2 = -8; \\ U_6 &= -X_2^2 - Y_0^2 + 2 = -3^2 - 3^2 + 2 = -16; \\ U_8 &= -X_0^2 - Y_1^2 + 2 = -2^2 - 2^2 + 2 = -6; \\ U_2 &= -X_0^2 - Y_2^2 + 2 = -2^2 - 4^2 + 2 = -18 \end{aligned}$$

The coordinates of the gradient field at a given point

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{U_6 - U_4}{X_2 - X_1} = \frac{-16 - (-8)}{3 - 1} = -4; \\ \frac{\partial u}{\partial y} &= \frac{U_2 - U_8}{Y_2 - Y_1} = \frac{-18 - (-6)}{4 - 2} = -6 \end{aligned}$$

corresponding coordinates defined by the formula (1).

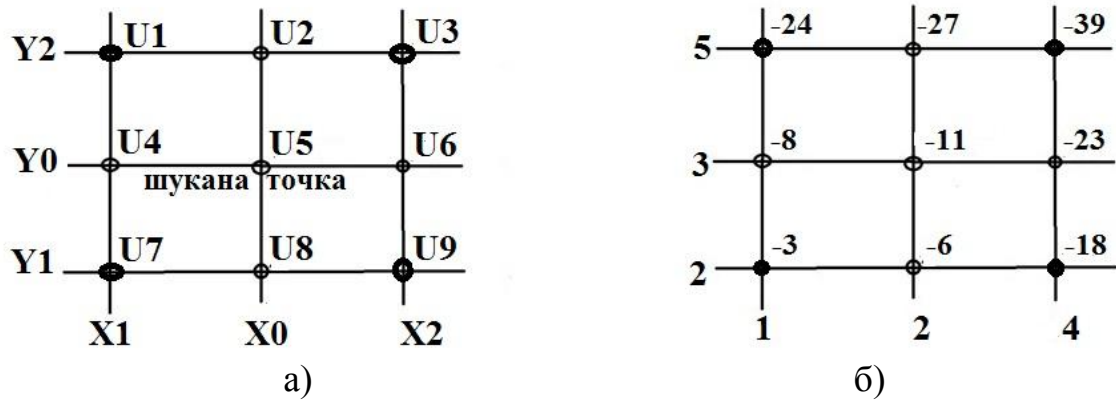


Fig.1. Detail of a scalar field on a rectangular grid:
a) coordinate grid; b) the coordinates on the non-uniform grid

In the case of irregular grid, eg $X_0 = 2, 3 = Y_0, X_1 = 1, X_2 = 4, Y_1 = 2, Y_2 = 5$ (Fig. 1b)

$$\begin{aligned}
 U_4 &= -X_1^2 - Y_0^2 + 2 = -1^2 - 3^2 + 2 = -8, \\
 U_6 &= -X_2^2 - Y_0^2 + 2 = -4^2 - 3^2 + 2 = -23, \\
 U_8 &= -X_0^2 - Y_1^2 + 2 = -2^2 - 2^2 + 2 = -6, \\
 U_2 &= -X_0^2 - Y_2^2 + 2 = -2^2 - 5^2 + 2 = -27.
 \end{aligned}$$

The coordinates of the gradient field at a given point

$$\begin{aligned}
 \frac{\partial u}{\partial x} &= \frac{U_6 - U_4}{X_2 - X_1} = \frac{-23 - (-8)}{4 - 1} = -5, \\
 \frac{\partial u}{\partial y} &= \frac{U_2 - U_8}{Y_2 - Y_1} = \frac{-27 - (-6)}{5 - 2} = -7,
 \end{aligned}$$

do not meet location, the analytical formula (1).

Where a scalar field defined on the set of unordered points, this method did not apply.

Therefore there is a need to develop a method for determining the gradient of the discrete scalar field given arbitrarily.

Analysis of recent research and publications. In papers [2,3] in determining the contours of the images to color images gradient calculated by difference equations:

$$\begin{aligned}
 G_X &= (U_7 + 2U_8 + U_9) - (U_1 + 2U_2 + U_3), \\
 G_Y &= (U_3 + 2U_6 + U_9) - (U_1 + 2U_4 + U_7).
 \end{aligned} \tag{3}$$

Both of these values are easily calculated using convolution image of two masks Sobel [2, 3]

$$G_Y = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} * U \quad G_X = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} * U$$

Also often used masks operator Previtta [3]

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} i \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

and Scharra [3]

$$\begin{bmatrix} 3 & 10 & 3 \\ 0 & 0 & 0 \\ -3 & -10 & -3 \end{bmatrix} i \begin{bmatrix} 3 & 0 & -3 \\ 10 & 0 & -10 \\ 3 & 0 & -3 \end{bmatrix}$$

Determine the value of the gradient of the scalar field $u = -x^2 - y^2 + 2$ the point (2,3), provided that it asked the right to non-uniform grid (fig. 1, b) using the abovementioned mask.

Sobel mask: $G_X = -60$, $G_Y = -84$.

Mask Previtta: $G_X = -45$, $G_Y = -63$.

Mask Scharra: $G_X = -240$, $G_Y = -336$.

Apparently, the coordinates do not correspond to the gradient values calculated analytically by formulas (1).

Formulation of the article purposes. The article is to develop a method for determining the gradient of the discrete scalar field defined on unordered sets of points.

Main part. According to [1] The rate of change of the scalar field in a given direction is equal to the scalar product of the gradient field on unit vector of direction.

$$\frac{du}{ds} = \text{grad } u \cdot \tau$$

In our case, the scalar field $u(r(x, y))$ (fig. 2) in determining the gradient at the point r_0

$$du = u_1 - u_0 ; ds = |r_1 - r_0| ; \tau = \frac{r_1 - r_0}{ds}, \quad (4)$$

where du – changing the scalar field u in a given direction;

ds – the length of the field for a given direction;

τ – unit vector of direction.

Algorithm for the scalar field gradient $g u$, given unordered set to r , to the point r_0 , based on the above-mentioned statement and looks like.

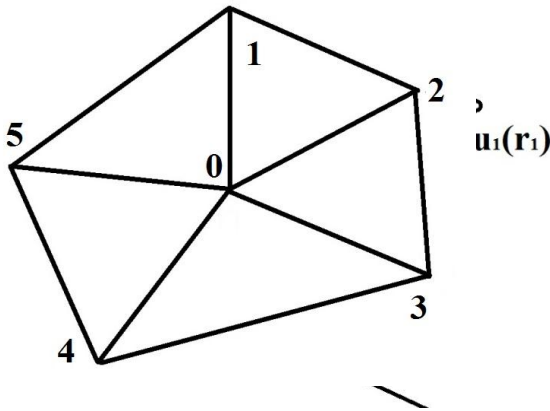
1. For a given set point scalar field do Delaunay triangulation.
2. Determine the triangles incident to top r_0 (Fig. 3).
3. Fold the matrix to determine the gradient

$$B = \begin{pmatrix} u_1 - u_0 \\ |r_1 - r_0| \\ u_2 - u_0 \\ |r_2 - r_0| \\ \dots \\ u_5 - u_0 \\ |r_5 - r_0| \end{pmatrix} \quad A = \begin{pmatrix} r_1 - r_0 \\ |r_1 - r_0| \\ r_2 - r_0 \\ |r_2 - r_0| \\ \dots \\ r_5 - r_0 \\ |r_5 - r_0| \end{pmatrix}$$

Collected system of linear equations

$$A * g = B.$$

4. The solution of this system and will coordinate gradient vector g at the point r_0 .



Since the matrix A is mostly rectangular, then the solution of this system of equations using QR-decomposition [4].

Test calculations showed that the largest error in determining the gradient occurs on the outer mesh tops (tops of the ribs belonging to one

triangle). Therefore, these peaks gradient determined by interpolation surface 2 order by the values in the next 10 Internal tops.

Table 1 shows the error of determining the gradient of scalar fields developed various ways and means of MATLAB in relation to the classical method. Testing conducted on an occasional monotone grid. The top number indicates the error in the x, bottom - on y. Testing of both methods on rectangular grids of regular errors showed approximately zero.

Conclusions. A method for determining the gradient of the discrete scalar field defined on a flat set of unordered points. The analysis accuracy compared to existing methods.

Table 1

The error determining the gradient in different ways

Scalar field	The gradient of the classical method	Error, %	
		Means Matlab	A method
$u = c * r$ $c = (2, 1)$ – constant vector	$grad u = 2i + 1j$	12	0
		22	0
$u = x^2 - y^2$	$grad u = 2xi - 2yj$	3.5	1.2
		2.4	0.8

$u=5x+2y+y*\cos(x)$	$\mathbf{grad\ u}=(5-y*\sin(x))\mathbf{i} + (2+\cos(x))\mathbf{j}$	4.6 10.8	0.6 0.8
---------------------	--	-------------	------------

Literature

1. Гольдфайн И.А. Векторный анализ и теория поля / И.А.Гольдфайн. – М.: Государственное издательство физико-математической литературы, 1962. – 132 с.
2. Гонсалес Р. Цифровая обработка изображений в среде MATLAB / Р. Гонсалес, Р. Вудс, С. Эддинс. – М.: Техносфера, 2006. – 616 с.
3. Алгоритмы выделения контуров изображений [Электронный ресурс] – Режим доступа: <https://habrahabr.ru/post/114452/> .
4. Потемкин В.Г. Вычисления в среде MATLAB / В.Г. Потемкин. – М.: ДИАЛОГ-МИФИ, 2004. – 720 с.