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ANALYTICAL CONDITIONS FOR THE FORMATION OF ISOTROPIC LINES FOR CONSTRUCTION OF MINIMAL SURFACES

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Analytical dependencies for finding parametric equations of spatial isotropic lines are obtained. Analytical description of minimal surfaces in complex space made of isotropic lines as lines of a translation net.

Key words: isotropic line, minimal surface, arc differential of a curve, mean curvature of surface.

Formulation of the problem. The investigation of methods for finding parametric equations of isotropic lines of zero length is due to the problem of analytical description of minimal surfaces. The use of minimal surfaces in the design of surfaces of technical forms and architectural designs has the benefits of practical content. The tension at each point of the minimum surface is a constant value. Shells that have the geometric form of minimal surfaces, have an architectural expression, can overlap complex plans without the formation of geometry ruptures with sources of stress [1, p. 152]. Condition of equality of zero value of the average curvature H The minimum surface at all its points is a necessary condition for the minimum area of the compartment of the surface, limited by a plane or spatial curve (contour) on this surface.

Setting the minimum surface function $z = z(x;\ y)$, J. Lagrange one of the first to conclude that the function $z = z(x;\ y)$ must satisfy the differential equation of the Euler-Lagrange [2, p. 683] in partial derivatives, which in the general case is not integrable. One of the current trends in analytical description of minimal surfaces is the finding of boundary conditions for the construction of minimal surfaces inserted into an open sphere [3]. Known are studies on geometric modeling of composite materials that form a cellular structure [4]. The architecture of these composite materials is based on periodic minimal surfaces, which allows them to minimize the effects of stress concentration, provide resistance to damage and reduce vibration [4].

The problem of simplifying the analytical description of minimal surfaces and obtaining their parametric equations, starting with the work of S. Lie, is realized with the help of methods of the theory of functions of a complex variable [2, p. 685].

Analysis of recent research and publications. To find the analytic description of the minimal surfaces using the functions of a complex variable, it is necessary to find parametric equations of isotropic lines of zero length. Simulation of spatial isotropic curves with quaternions in space R^4, considered in the paper [5]. In the dissertation research [6] an analytical description of isotropic curves according to formulas is found H. Schwarz on the basis of the spatial curve lying on the cylinder surface and on the basis of the slope curve. But the use of Schwarz's formulas for finding parametric equations of isotropic lines is due to the integration of complex expressions and is possible only in some cases. In the dissertation study [6] and in article [7], separate cases of analytical description of isotropic lines were considered on the basis of equations of a plane imaginary curve of the form: $x(t)= f_1(t)+i\cdot f_2(t);\quad y(t)= f_1(t)-i\cdot f_2(t)$, where $f_1(t)$ i $f_2(t)-$ function of a real or complex variable. The work [8] of the authors of this article is devoted to the problem of analytical description of isotropic lines on the basis of a plane curve given by functions of a natural parameter. In spite of the variety of known methods for generating parametric equations of isotropic lines, simplifying their analytical description remains an important problem in the simulation of a continuous frame of minimal surfaces.

Formulating the goals of the article. To determine the analytical relations of the formation of spatial isotropic lines from the condition of equalization of zero of their arc differential. Using the functions of a complex variable, we find the parametric equations of isotropic lines and the corresponding minimal surfaces.

Main part. Let us consider the statement in which analytical relations of the formation of spatial isotropic lines of zero length with the help of functions of a complex variable are determined.

Assertion. *Let the functions* $u=u(t);\ v=v(t)$ *satisfying equality:* $u^2(t)+v^2(t)=1$. *Then the spatial curve given by the equations:*

$$x=\int\left(u+\frac{1}{2(v-u)}\right)dt;\quad y=\int\left(v-\frac{1}{2(v-u)}\right)dt;\quad z=\frac{i}{\sqrt{2}}\int\frac{dt}{v-u} \tag{1}$$

is an isotropic line of zero length.

Proving Let's write the equation of the isotropic line in the formi:

$$x(t)=p(t)-i\cdot f(t);\quad y(t)=q(t)+i\cdot f(t);\quad z=z(t), \tag{2}$$

where $p(t),\ q(t),\ f(t)-$ differentiated on some function interval, $i-$ imaginary unit. Then the expression of the differential of the arc of the isotropic line has the form:

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=\left(\frac{dp}{dt}\right)^2-2i\frac{dp}{dt}\frac{df}{dt}-\left(\frac{df}{dt}\right)^2+\left(\frac{dq}{dt}\right)^2+2i\frac{dq}{dt}\frac{df}{dt}-$$

$$-\left(\frac{df}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=\left(\frac{dp}{dt}\right)^2+\left(\frac{dq}{dt}\right)^2+2i\frac{df}{dt}\left(\frac{dq}{dt}-\frac{dp}{dt}\right)-2\left(\frac{df}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2.$$

The arc differential of the isotropic line is zero when the conditions are fulfilled simultaneously:

$$\left(\frac{dp}{dt}\right)^2+\left(\frac{dq}{dt}\right)^2=1;\quad 2i\frac{df}{dt}\left(\frac{dq}{dt}-\frac{dp}{dt}\right)=-1;\quad -2\left(\frac{df}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=0.$$

From the last equations we express: $\frac{df}{dt}=-\frac{1}{2i\left(\frac{dq}{dt}-\frac{dp}{dt}\right)}$ та $\frac{dz}{dt}=\sqrt{2}\frac{df}{dt}$,

then we will get:

$$\frac{dx}{dt}=\frac{dp}{dt}-i\frac{df}{dt}=\frac{dp}{dt}+\frac{1}{2\left(\frac{dq}{dt}-\frac{dp}{dt}\right)};$$

$$\frac{dy}{dt}=\frac{dq}{dt}+i\frac{df}{dt}=\frac{dq}{dt}-\frac{1}{2\left(\frac{dq}{dt}-\frac{dp}{dt}\right)};\quad \frac{dz}{dt}=\sqrt{2}\frac{df}{dt}=\frac{i}{\sqrt{2}\left(\frac{dq}{dt}-\frac{dp}{dt}\right)}. \qquad (3)$$

Integrating equation (3) and introducing a replacement $u(t)=\frac{dp}{dt}$; $v(t)=\frac{dq}{dt}$, we obtain the parametric equation (1) of the isotropic line.

The statement is proven.

Example. We find parametric equations of an isotropic line by formulas (1) for functions $u(t)=\sin(kt); v(t)=\cos(kt)$, where $k\in R$.

Then, using the formulas (1), we obtain the parametric equations of the imaginary isotropic line:

$$x(t)=\frac{1}{k}\left[\frac{1}{\sqrt{2}}\operatorname{Arth}\left(\frac{1+tg\left(\frac{kt}{2}\right)}{\sqrt{2}}\right)-\cos(kt)\right];$$

$$y(t)=\frac{1}{k}\left[\frac{-1}{\sqrt{2}}\operatorname{Arth}\left(\frac{1+tg\left(\frac{kt}{2}\right)}{\sqrt{2}}\right)+\sin(kt)\right];\quad z(t)=\frac{i}{k}\operatorname{Arth}\left(\frac{1+tg\left(\frac{kt}{2}\right)}{\sqrt{2}}\right). \qquad (4)$$

We perform the functions of a complex variable (4) substitution: $t=u+i\cdot v$. Separating the real and imaginary part, we obtain the equation of the minimal surface:

$$X(u,v)=-\frac{\cos(ku)\operatorname{ch}(ku)}{k}+$$
$$+\frac{1}{4\sqrt{2}k}\left[\ln\left((1+m(u;v))^2+n(u;v)\right)-\ln\left((1-m(u;v))^2+n(u;v)\right)\right];$$
$$Y(u,v)=\frac{\sin(ku)\operatorname{ch}(kv)}{k}+ \qquad (5)$$
$$+\frac{1}{4\sqrt{2}k}\left[\ln\left((1-m(u;v))^2+n(u;v)\right)-\ln\left((1+m(u;v))^2+n(u;v)\right)\right];$$
$$Z(u,v)=\frac{1}{2k}\left[\operatorname{arctg} j(u;v)-\operatorname{arctg} w(u;v)\right],$$

where:

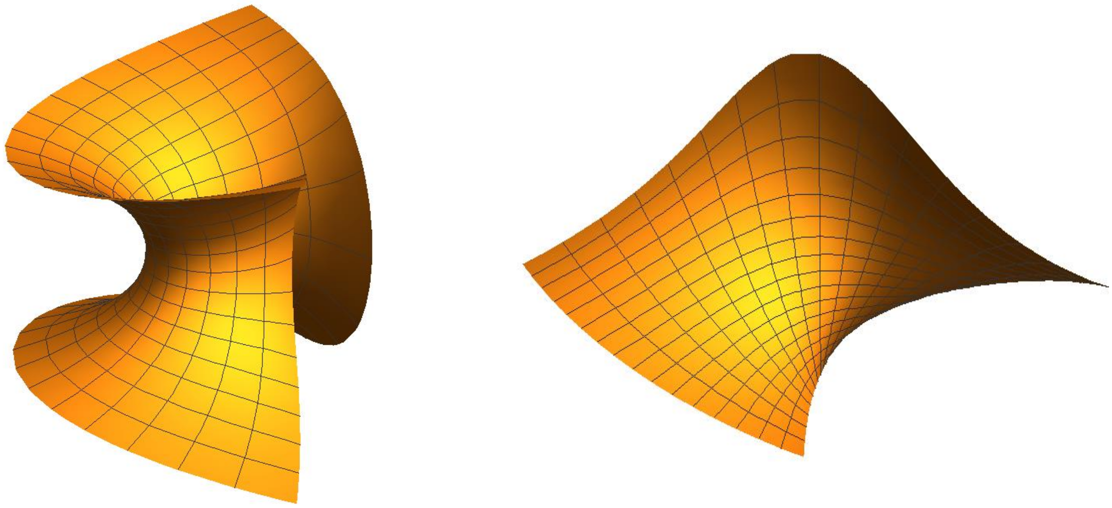
$$m(u;\,v)=\frac{1}{\sqrt{2}}\left(1+\frac{\sin ku}{\cos ku+\operatorname{ch} kv}\right);\ n(u;\,v)=\frac{\operatorname{sh}^2(kv)}{2(\cos ku+\operatorname{ch} kv)^2};$$
$$j(u;v)=\frac{\sqrt{2}\operatorname{sh}(kv)}{\left(-2+\sqrt{2}\right)(\cos ku+\operatorname{ch} kv)+\sqrt{2}\sin ku}; \qquad (6)$$
$$w(u;v)=\frac{\sqrt{2}\operatorname{sh}(kv)}{\left(2+\sqrt{2}\right)(\cos ku+\operatorname{ch} kv)+\sqrt{2}\sin ku}.$$

and the attached minimal surface:

$$X^*(u,v)=\frac{\sin(ku)\operatorname{sh}(kv)}{k}+\frac{1}{2\sqrt{2}k}\left(\operatorname{arctg} w(u;v)-\operatorname{arctg} j(u;v)\right);$$
$$Y^*(u,v)=\frac{\cos(ku)\operatorname{sh}(kv)}{k}+\frac{1}{2\sqrt{2}k}\left(\operatorname{arctg} j(u;v)-\operatorname{arctg} w(u;v)\right); \qquad (7)$$
$$Z^*(u,v)=\frac{1}{4k}\left[\ln\left((1+m(u;v))^2+n(u;v)\right)-\ln\left((1-m(u;v))^2+n(u;v)\right)\right]$$

In the parametric equations (7) expressions $m(u;\,v), n(u;\,v), j(u;\,v), w(u;\,v)$ are determined from (6).

Fig. 1 (a, b) shows the minimal surfaces constructed by equations (5), (7) with $k=1;\ u\in\left[-\frac{\pi}{5};...\frac{\pi}{5}\right];\ \ v\in[-1;...1]$.



a) b)

Fig. 1. Bins of minimum surfaces:

a) minimal surface, constructed by equations (5) at $k=1$;

б) Attached min. surface constructed by equations (7) at $k=1$.

Conclusions. The analytic dependences of the formation of spatial isotropic lines defined in this paper allow us to find the parametric equations of the minimal surfaces. In particular, an analytical description of isotropic lines can be found on the basis of plane curves given by parametric equations of arc length s.

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АНАЛИТИЧЕСКИЕ УСЛОВИЯ ОБРАЗОВАНИЯ ИЗОТРОПНЫХ ЛИНИЙ ДЛЯ ПОСТРОЕНИЯ МИНИМАЛЬНЫХ ПОВЕРХНОСТЕЙ

Пилипака С.Ф., Муквич Н.Н., Федорина Т.П.

Получены аналитические зависимости для нахождения параметрических уравнений пространственных изотропных линий. Аналитическое описание минимальных поверхностей осуществлено в комплексном пространстве с изотропными линиями в качестве линий сети переноса.

Ключевые слова: изотропная линия, минимальная поверхность, дифференциал дуги кривой, средняя кривизна поверхности.

АНАЛІТИЧНІ УМОВИ УТВОРЕННЯ ІЗОТРОПНИХ ЛІНІЙ ДЛЯ ПОБУДОВИ МІНІМАЛЬНИХ ПОВЕРХОНЬ

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У роботі визначено аналітичні залежності утворення просторових ізотропних ліній за допомогою функцій комплексної змінної. Знайдено параметричні рівняння ізотропних ліній та мінімальних поверхонь у комплексному просторі з твірними ізотропними лініями переносу.

Ключові слова: ізотропна лінія, мінімальна поверхня, диференціал дуги просторової кривої, середня кривина поверхні.