УДК 514.18

CLOSED SMOOTH CURVES CONSTRUCTION WITH THE GAUSSIAN INTERPOLATING POLYNOMIAL

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The paper discusses two algorithms for constructing smooth closed curves using the Gaussian interpolation function.

In the work of an engineer, there is a constant need to construct geometric objects that have given properties. Components of these objects can be closed curves. In complex technical problems, in addition to closedness, an additional condition of smoothness at all points of the curve may arise. Therefore, the relevance of constructing such smooth closed curves is relevant.

Gaussian interpolation polynomials, being based on exponential functions, have several advantages over algebraic interpolation polynomials. There is no oscillation that is typical for algebraic polynomials when interpolating larger data sets. Also, using the additional coefficients in the argument of each term, it is possible to control the function when solving any problem that both requires interpolation and implies some additional properties of the interpolant.

Parameterization of the interpolating Gaussian polynomial enables the construction of curves. To make curves closed, we add the condition: the point corresponding to the lowest value of the parameter must coincide with the point corresponding to the highest value. We call this point the gluing point.

However, until now, the use of Gaussian interpolation polynomial for constructing closed curves was limited due to the fact that the methods of constructing such a polynomial did not provide control over the derivatives at the interpolation points, and therefore over the tangent lines at the points of the parametric curves and the order of smoothness at the gluing point when constructing a closed curve.

In this paper, two fundamentally different methods of solving the problem of constructing a smooth closed curve with a Gaussian interpolation polynomial are proposed, each, however, with its own advantages and disadvantages. The first method involves setting the values of the derivatives at the points of the interpolation polynomial explicitly, and the second – the smooth closure of the curve at the point of gluing by means of mathematical optimization of the difference of the derivatives in the gluing pint without setting the derivatives explicitly.

The purpose of the research is to improve the construction of curvilinear contours using the Gaussian interpolation function by introducing the smoothness condition. Two different methods are proposed, and the results of their work are showcased.

Keywords: interpolation, Gaussian interpolation curve, curve smoothness, mathematical optimization.

Problems staging. In previous articles, the algorithm for constructing an interpolation function based on Gaussian curves was given. This algorithm in most cases has better accuracy compared to classical interpolation methods. In addition, since this function belongs to exponential curves, this makes it possible to differentiate it infinitely. In its existing form, the Gaussian interpolation function does not account for derivatives in points, however, it doesn't prohibit such control explicitly. To get rid of this drawback we can exploit the function's properties without reinventing the polynomial per se.

Analysis of recent research. The algorithm for constructing the Gaussian interpolation function was first introduced in [1]. The possibilities of using exponential interpolation for the construction of smooth non-closed objects were also shown there. The results of the analysis of the Gaussian interpolation function algorithm on elementary algebraic functions are given in [2]. It is shown that for most functions with an uneven frame, this function has better accuracy compared to the classical Lagrange polynomial. The work [3] is devoted to the analysis of the influence of the variable parameter on the appearance of the interpolation function. Methods of solving mathematical optimization problems are given in [4], in particular, the algorithm that was used in the implementation of the showcased curve construction algorithm.

Objectives of the study. The purpose of the study is to construct a smooth closed curve at the given points by either introducing additional equations taking into account the given derivatives at the interpolation points, or by optimizing the derivatives difference in the gluing point by means of mathematical optimization.

Main part. The Gaussian interpolation polynomial has the form:

$$P(x) = \sum_{i=1}^{n} \gamma_i e^{-\alpha_i (x-x_i)^2}$$

where x_i – interpolation points (i = 1..n), α_i – positive coefficients in the argument of the exponent, γ_i – polynomial coefficients.

Usually, to find the coefficients of the polynomial γ_i, a system of linear equations $P(x_i) = y_i$ is formed. This system has n linear equations, n variables γ_i, respectively, if all x_i are unique, such a system has one and only one solution.

Analytical method of constructing a smooth closed curve by Gaussian interpolation polynomial.

The first method of constructing a closed curve with a Gaussian interpolation polynomial consists of enriching the system of equations of the form $P(x_i) = y_i$ with equations of the form:

$$\frac{dP(x_i)}{dx} = dy_i$$

where dy_i is the given polynomial derivative at the interpolation point y_i.

The derivative of each term of the Gauss interpolation polynomial:

$$e^{-\alpha_i(x-x_i)^2}$$

is a function:

$$-\alpha_i(2x - 2x_i)e^{-\alpha_i(x-x_i)^2}$$

Accordingly, the derivative of the polynomial will be the sum of the γ_i-weighted derivatives of each term.

Note that the task of finding optimal α_i is not considered in this context. For simplicity, we can assume that $\alpha_i = 1$.

So, in order to specify the derivatives at the points, you need to add the appropriate number of equations to the system. Unfortunately, it is not possible, as is usually done in the case of algebraic polynomials, to raise the degree of the polynomial and, accordingly, increase the number of its coefficients by a number sufficient for the system to remain defined when additional equations are added. Raising the degree, or more precisely, adding a term to the Gaussian interpolation polynomial, involves the introduction of new points of the interpolation frame x_i, even if these points will not strictly speaking be interpolating. In other words, there is no need for the polynomial to pass through these added points.

So, for example, for an interpolation polynomial that passes through two points (x_1, y_1), (x_2, y_2) and has the values of the derivatives dy_1, dy_2 defined at these points, we must construct a system of the following form:

$$\gamma_1 e^{-\alpha_1(x_1-x_1)^2} + \gamma_2 e^{-\alpha_2(x_1-x_2)^2} + \gamma_3 e^{-\alpha_3(x_1-x_3)^2} + \gamma_4 e^{-\alpha_4(x_1-x_4)^2} = y_1$$

$$\gamma_1 e^{-\alpha_1(x_2-x_1)^2} + \gamma_2 e^{-\alpha_2(x_2-x_2)^2} + \gamma_3 e^{-\alpha_3(x_2-x_3)^2} + \gamma_4 e^{-\alpha_4(x_2-x_4)^2} = y_2$$

$$-\alpha_1(2x_1 - 2x_1)e^{-\alpha_1(x_1-x_1)^2} + -\alpha_2(2x_1 - 2x_2)e^{-\alpha_2(x_1-x_2)^2} + -\alpha_3(2x_1 - 2x_3)e^{-\alpha_3(x_1-x_3)^2} + -\alpha_4(2x_1 - 2x_4)e^{-\alpha_4(x_1-x_4)^2} = dy_1$$

$$-\alpha_1(2x_2 - 2x_1)e^{-\alpha_1(x_2-x_1)^2} + -\alpha_2(2x_2 - 2x_2)e^{-\alpha_2(x_2-x_2)^2} + -\alpha_3(2x_2 - 2x_3)e^{-\alpha_3(x_2-x_3)^2} + -\alpha_4(2x_2 - 2x_4)e^{-\alpha_4(x_2-x_4)^2} = dy_2$$

It can be seen that the values of x_3 and x_4 appeared in the system, although they were not present in the initial formulation of the problem. These are non-interpolated frame values that do not force the polynomial to pass through the corresponding points y_3 and y_4. Actually, the system does not even contain such values, but adding the x_3 and x_4 allows us to formulate a polynomial of four terms to build a system of four equations.

When solving this system by any method, we will get the coefficients of the interpolation polynomial. Systems for polynomials with a larger number of interpolation points are solved similarly.

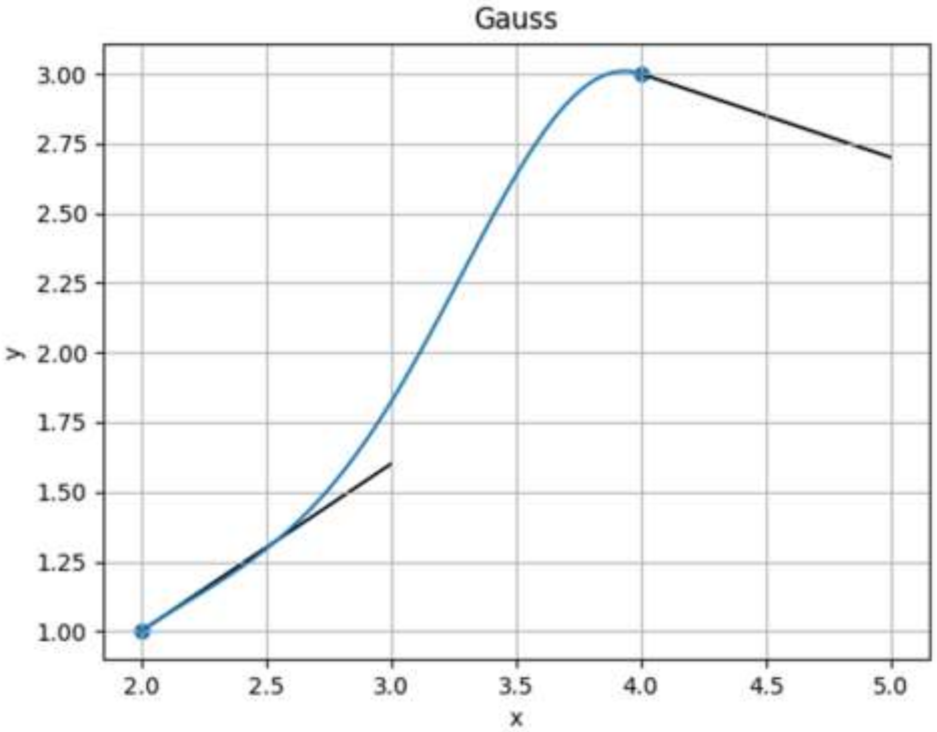

Figure 1. Interpolating polynomial by 2 points with 2 derivatives

Figure 1 shows an example of an interpolation polynomial that passes through two points (2,1), (4,3) and has derivative values of 0.6 and -0.3, respectively.

To construct a smooth closed curve by specifying the derivatives at the points of the Gaussian interpolation polynomial, it is necessary to define two polynomials dependent on the common parameter t: one for the x-coordinate: $P_x(t)$, the other for the y-coordinate: $P_y(t)$.

The set of points ($P_x(t)$, $P_y(t)$) will form a continuous curve. If the coordinates of the first and last point of the interpolation frame coincide, the curve will have a gluing point. Limiting this curve with a parameter between the first and last point will make the curve closed.

To ensure the smoothness of such a curve, it is enough to always add only two equations for each coordinate. One will set the derivative of the corresponding polynomial at the starting point, the other at the final point. When parametrically specifying a closed curve, the coordinates of the starting point must coincide with the coordinates of the final point. Under the condition of equivalence of the derivatives at the first and last point, the curve will be smooth.

The drawback of this method is that the additional non-interpolating points that had to be added affect the appearance of the function. For instance, figure 1 shows an example of an interpolation polynomial that passes through two points (2,1), (4,3) and has derivative values of 0.6 and -0.3 determined in them, respectively. Additional non-interpolating points of that polynomial were: $x_3 = 3$ and $x_4 = 5$

If you choose other additional points, the specified interpolation and differentiation properties of the polynomial will not change, but the general appearance of the polynomial will.

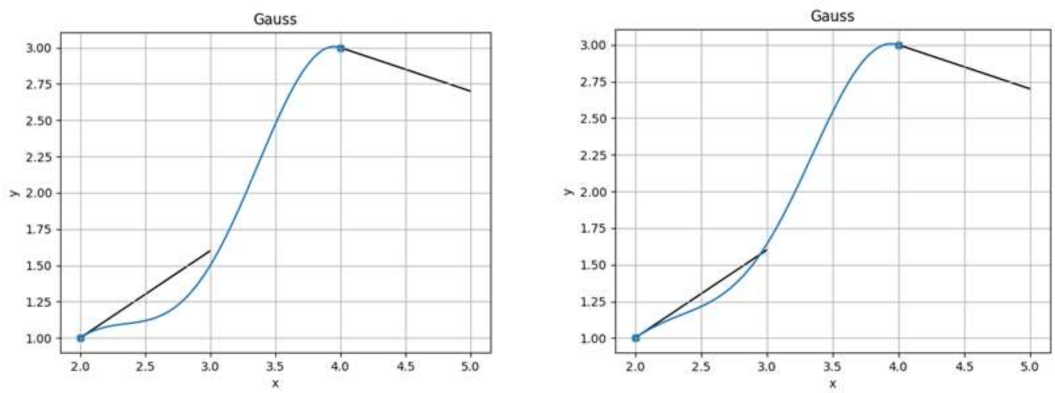

Figure 2. Interpolating polynomials with different added points

Figure 2 shows an example of the interpolating Gaussian polynomial with the same interpolation and differentiation properties at points as in Figure 1 but with different additional points ($x_3 = 0$ and $x_4 = 6$ on the left, $x_3 = 2.001$ and $x_4 = 4.001$ on the right).

Choosing the values of additional points of the Gaussian interpolation polynomial with given derivatives at the points is a separate problem that is beyond the scope of this paper. Probably, as with the coefficients α_i, the choice of specific values will depend on the context of the application of the interpolation function.

Unlike the terms of an algebraic polynomial, the terms of a Gaussian polynomial can be differentiated infinitely. Using this property, it is possible to provide any predetermined order of smoothness at the self-adjoint points by adding new pairs of equations for each new degree of smoothness.

Therefore, this method of constructing closed smooth curves by Gaussian interpolation polynomial allows us to set any given order of smoothness in the gluing point by adding a pair of linear equations for every degree of smoothness. This is also possible with algebraic polynomials, but the advantage of Gaussian interpolation is that it doesn't tend to oscillate when adding new points or derivatives at points unlike algebraic polynomials do.

The disadvantages of this method, however, are that, firstly, it is necessary to determine the derivatives at the gluing point in advance, and secondly, we also need to somehow determine additional non-interpolating points of the frame, which are not essential for the actual task of constructing a closed curve.

In addition, this method is orthogonal to the determination of coefficients α_i. Depending on the context of each specific application, this can be both an advantage and a disadvantage.

Numerical method of constructing a closed curve by Gaussian interpolation polynomial.

The second way of constructing a closed curve by the Gaussian interpolation polynomial is to numerically optimize the difference between the derivatives of the interpolation polynomial at the first and last point precisely in the space of α_i coefficients.

The target function for optimization is the sum of squared differences of the derivatives for each variable at the first and last point of the curve constructed by the Gaussian interpolation polynomial. Also, considering that the optimization is conducted by numerical methods anyway, which implies non-zero numeric error, we can safely use finite differences instead of derivatives.

With the target function established, we can use any numerical optimization algorithm to find the local optimum of the derivative at the first and last point difference. To showcase the obtained results, the Broyden-Fletcher-Goldfarb-Shanno algorithm was used, but the problem itself does not infer the use of any particular algorithm. The choice of the best mathematical optimization algorithm for each specific application problem of constructing a smooth closed curve goes beyond the objectives of this paper.

The result of the system can be seen in Figure 3.

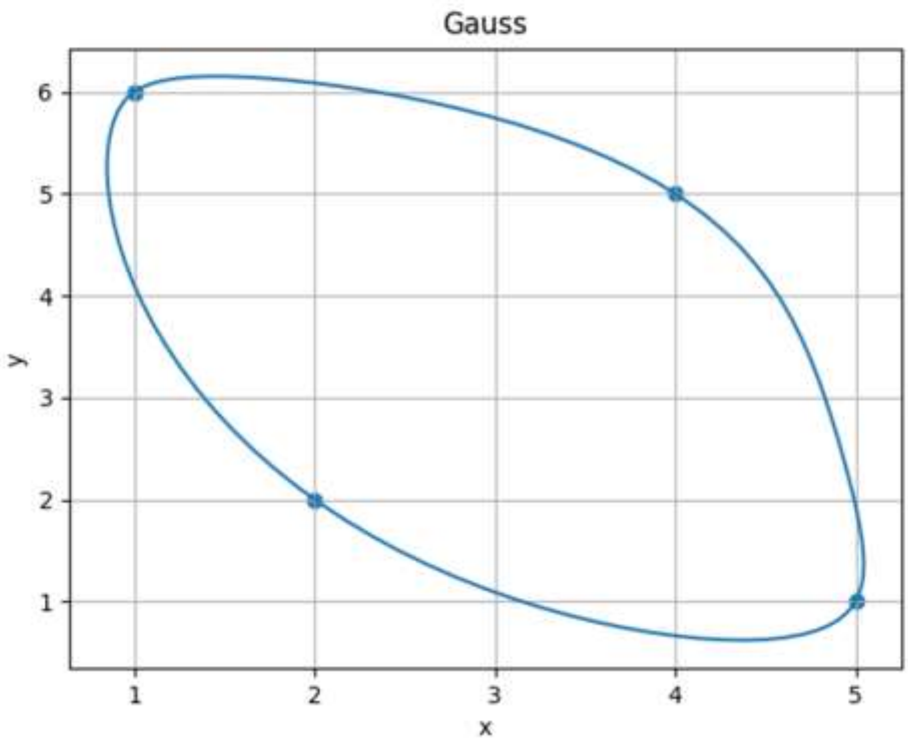

Figure 3. Closed smooth curve

Figure 3 shows a closed smooth curve constructed by a Gaussian interpolation polynomial with coefficients obtained by the numerical method of minimizing the sum of squares of the finite differences of the derivatives in the first (1, 6) and the last, which coincides with the first.

Unlike the previous method, this method does not require the introduction of additional non-interpolating points. It also does not leave the issue of determining the coefficients α_i without consideration. But it requires the solution of a system of linear equations at each iteration of the minimization algorithm, which is a problem when the dimensionality of the system increases, not only because of the corresponding increase in the number of calculations when solving each system but also because of the increase in the dimension of the optimization space and the number of iterations, respectively.

However, for constructing curves based on a small number (up to a dozen) of interpolation points, this method is relatively undemanding in terms of computing power.

Conclusions. Two methods of constructing a smooth closed curve with a Gaussian polynomial were proposed. Both methods are effective. Each can be applied in one or another context, depending on the additional conditions imposed on the task. Both methods can be developed further and adapted to solve application-specific problems.

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ПОБУДОВА ЗАМКНЕНИХ ГЛАДКИХ КРИВИХ ІНТЕРПОЛЯЦІЙНИМ ПОЛІНОМОМ ГАУСА

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У статті розглянуто два алгоритми побудови гладких замкнених кривих за допомогою інтерполяційної функції Гауса.

У роботі інженера постійно виникає необхідність конструювати геометричні об’єкти, які мають задані властивості. Компонентами цих об’єктів можуть бути замкнені криві. У складних технічних задачах, окрім замкненості, може виникати і додаткова умова про гладкість у всіх точках

кривої. Тому актуальність побудови таких гладких замкнених кривих є актуальною.

Інтерполяційний поліном Гауса, побудований на базі експоненціальних функцій, має ряд переваг над алгебраїчними інтерполяційними поліномами. При додаванні нових точок каркасу не виникає осциляцій, а завдяки додатковим коефіцієнтам при аргументі кожного члена, можна керувати виглядом функції при розв'язанні задачі інтерполяції.

За допомогою параметризації функції Гауса вдалось проводити інтерполяцію замкнених кривих. Алгоритм передбачав завдання першої і останньої точки, які співпадали. Надалі цю точку будемо називати точкою склеювання.

Втім, дотепер застосування інтерполяційного полінома Гауса для побудови замкнених кривих було обмежено через те, що способи побудови такого полінома не передбачали контролю за похідними в точках інтерполяції, а значить і за дотичними лініями в точках параметричних кривих, а значить і за порядком гладкості у точці склеювання при побудові замкненої кривої.

В даній роботі пропонується два принципово різних способи вирішення задачі побудови гладкої замкненої кривої інтерполяційним поліномом Гауса, кожен, втім, зі своїми перевагами, обмеженнями і недоліками. Перший передбачає задання значень похідних у точках інтерполяційного полінома, другий – гладке замикання кривої у точці склеювання без явного контролю за похідними.

Метою досліджень є удосконалення способу побудови криволінійних обводів за допомогою інтерполяційної функції Гауса з урахуванням умов гладкості. У статті пропонується два підходи до розв'язання цієї задачі. Наводяться результати роботи системи, що покращує сприйняття матеріалу.

Ключові слова: інтерполяція, інтерполяційна крива Гауса, гладкість кривої, математична оптимізація.

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