УДК 514.18

RECONSTRUCTION OF A CURVE BY ITS POINT CLOUD USING GAUSSIAN POLYNOMIAL

Tarnavski Yu.A., Ph.D.,
tarnavski.yu @gmail.com, ORCID: 0000-0003-3226-3107
Mykhailova I.Yu., Ph.D.,
i.mykhailova@kpi.ua, ORCID: 0000-0003-1083-2815
Kaleniuk O.S., Ph.D.,
o.kaleniuk@kpi.ua, ORCID: 0009-0009-3141-4840
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"(Ukraine)

The paper describes the prospects of using a specific type of Gaussian polynomial function for the reconstruction of a continuous and regular surface by its point cloud.

Restoring the curves and surfaces behind their point clouds is an important sub-problem for 3D scanning-based modeling. Also, a similar problem arises when converting surfaces from boundary representation to triangle mesh. For the latter, it is critically important to restore the orientation of the surface if the initial data does not contain such information, or contains false information.

The Gaussian interpolation polynomial certainly goes to zero, while the argument of the function goes to plus- and minus-infinity at any configuration of the point cloud. Thus, if in a 3-dimensional space, we define the point cloud surface as an isosurface of some predefined positive value, for example, 1, and consider the surface oriented in the direction of isolines with a lower value, the surface constructed by a generalized interpolating Gaussian polynomial should always appear to be oriented correctly, i.e. the space bounded by the surface will always be closed, and the closure will contain the inner side of it. Moreover, the isosurface itself will be continuous and regular since the Gaussian functions that constitute the polynomial are infinitely differentiable.

Despite these useful properties, the interpolation Gaussian polynomial has not been used in point cloud-based curve restoration. The purpose of this research is to assess the practical applicability of Gaussian interpolation in the context of restoring curves and surfaces by their point clouds.

The challenge of such application lies not in obtaining the correct differential properties of the resulting curves and surfaces but in establishing their topological properties, namely connectivity, and their resulting shapes. The resulting shape of the curve or surface obtained by Gaussian interpolation should somehow correspond to the implied surface or curve behind the point cloud.

Keywords: interpolation, Gaussian polynomial, regular surfaces, point cloud, mathematical optimization, geometrical modeling, modeling.

Formulation of the problem. Let there be a point cloud consisting of *M* *N*-dimensional points P_i, $i = 1..M$. Each point represents a specific point of a surface we want to restore. We presume that the curve in question is continuous and regular.

We want to define an *N*-ar function, namely a Gaussian polynomial, that will be interpolating for some predefined positive value *C* defined in each point of the point cloud. The isosurface of this polynomial will be the target curve we're trying to restore.

Mathematically, the curve is guaranteed to be regular, however, we don't know whether its shape of connectivity is applicable to real-world applications. That's what we're establishing with a series of computational experiments.

Analysis of recent research and publications. The algorithm for constructing the unary Gaussian interpolation function was introduced in [1]. In Implicit Curves And Surfaces Modeling With Pseudogaussian Interpolation [2] we have shown that the pseudo-Gaussian interpolation could be used for implicit surfaces and curves representation. And in Closed Smooth Curves Construction With The Gaussian Interpolating Polynomial [3] we have shown that Gaussian interpolation could be used for building continuous, although not regular, curves with predefined degrees of continuity.

Also, surface reconstruction from a cloud of points in the form of a Gaussian sum, which is not, strictly speaking, an interpolating function, is already a popular approach. Thus, in the work of [4] Spherical Gaussian Point Cloud Representation is used not only to restore the shape of an object from the point cloud but also to simplify its registration for other similar objects. Prior to that, Spherical Gaussians were used to calculate illumination when rendering [5]. The use of approximating spherical Gaussians was first proposed in 2006 [6].

Formulating the purposes of the article. The purpose of the study is to establish whether restoring surfaces from point clouds by means of Gaussian interpolation is practically applicable. For this, a series of experiments are conducted, and the conditions in which the shape and connectivity of the resulting surface in 3D or curve in 2D are satisfactory are analyzed.

Main part. The study of Gaussian polynomial interpolation in the context of surface modeling in 3D is best to begin with modeling curves in in 2D. So, for example, if the reconstruction of a curve by its point cloud already shows the fundamental limitations of the method, or its computational problems, extending the method by adding another dimension should already seem impractical.

So, for practical consideration, we narrowed the problem of surface restoration down to constructing a smooth continuous correctly oriented curve that passes through a finite set of points.

For experimentation, we choose specifically a set of points obtained by randomizing the position of uniformly distributed points of a circle with radius 1. Randomization is carried out in a square with a side equal to a third of the length of the sector between two points in radians, with a normal random distribution.

The reconstructed curve is expected to generally follow the contour of the randomized circle from which it was constructed, with apparent deviations within the intervals of randomization. (Figure 2)

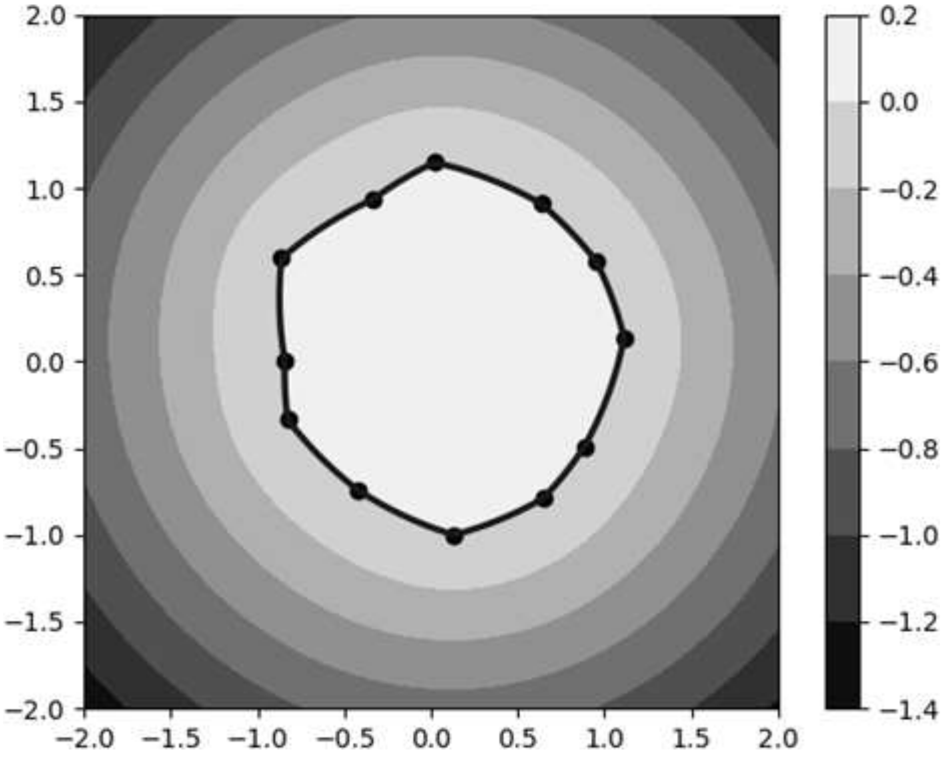

Figure 2. An example of an implicit curve built upon a test point set

To simplify the illustrative material, hereinafter we show the same set of 12 points, although in the course of the study, point clouds with different numbers of points and different randomization schemes were considered.

The construction of a correctly oriented implicit curve by an interpolating Gaussian polynomial.

An N-ar Gaussian polynomial G that interpolates a set of M N-dimensional points P_i, $i = 1..M$ has this formula:

$$G(x) = \sum_{i=1}^{M} \gamma_i e^{-a_i |x - P_i|}.$$

Here P_i – interpolation points ($i = 1..M$), α_i – positive coefficients in the argument of the exponent, γ_i – polynomial coefficients.

For any specific interpolating polynomial, the coefficients could be obtained by solving the system of equations:

$$\begin{cases} \sum_{i=1}^{M} \gamma_i e^{-a_i|P_1-P_i|} = C \ ; \\ \sum_{i=1}^{M} \gamma_i e^{-a_i|P_1-P_2|} = C \ ; \\ \qquad\qquad \cdots \\ \sum_{i=1}^{M} \gamma_i e^{-a_i|P_M-P_i|} = C \ . \end{cases}$$

The system becomes linear if we only solve it for γ_i with already predefined $\alpha_{i.}$ Obtaining the coefficients α_i that improve the quality of interpolation in one sense or another is, however, an open problem.

If, for instance, we assume that $a_i = 1$ for all i, then the curve generated on a predefined point cloud does not resemble a circle, but instead breaks up into several thin islands (Figure 3).

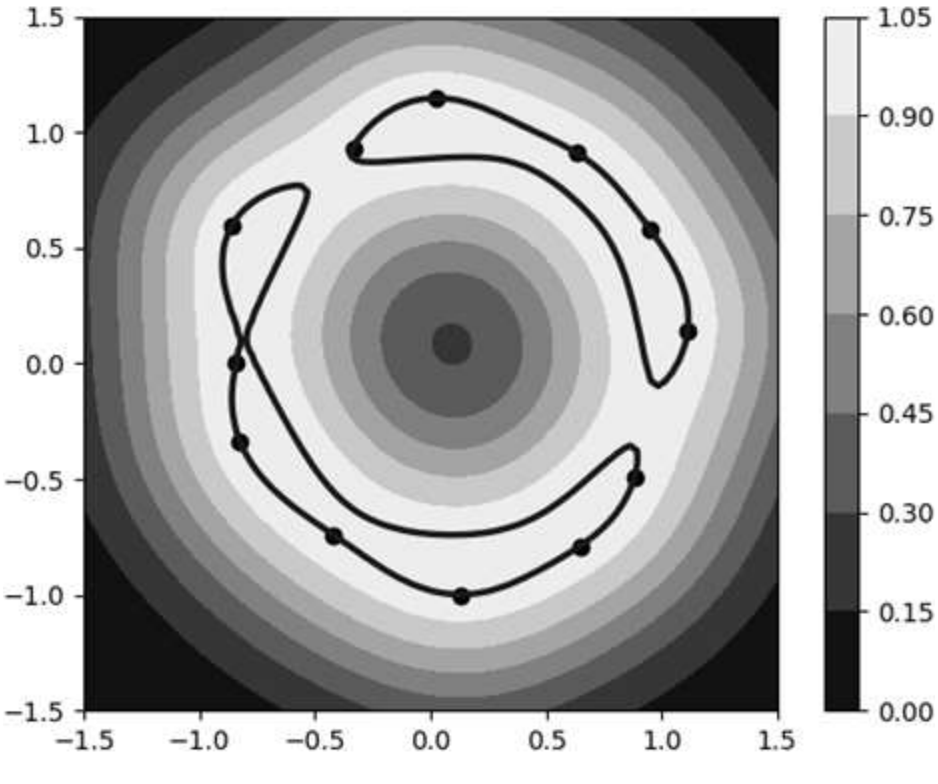

Figure 3. An example of a curve built with an unmotivated choice of a_i

In the paper [smooth gauss], a proposition to calculate a_i coefficients by optimizing a predetermined target function was suggested. It is expected that by choosing a target function of one kind or another, we will have control over the coefficients of the interpolation polynomial and, accordingly, the shape of the implicit curve that we are building.

In this work, we investigated the following target functions.

1. The total square error in the control points given by the same sample of points in the circle with the same randomization, but shifted by half the angle of the unit segment. It was expected that the addition of control points would allow for the selection of coefficients that would make the outline more similar to the outline of a circle, but expectations were not met. (Figure 4)

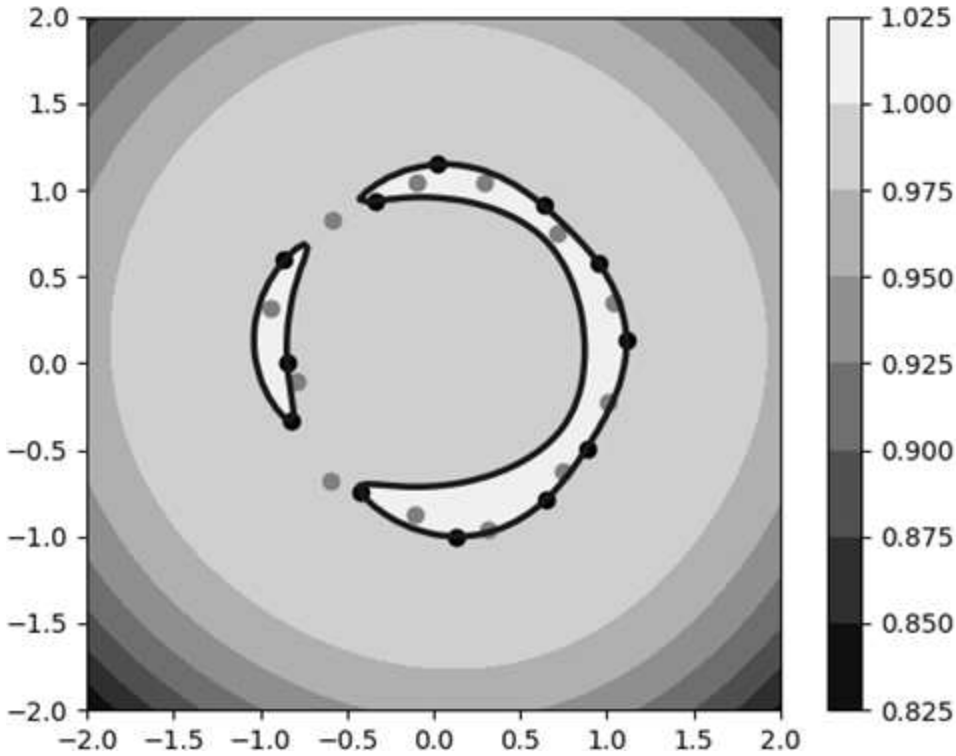

Figure 4. An example of a_i optimization by fitting the control points

None of the variants of point cloud randomization with the specified target function resulted in a single-linked curve. All the experiments led to a set of disjointed islands.

2. The other target function in question was the total square curvature of the curve at the interpolation points. Similar to the previous assumption, it was expected that minimizing the square of curvature at points would achieve a more circle-like contour – a figure with the minimum possible integral of the square of curvature. However, these expectations also did not come true (Figure 5).

This time, the curve could have been constructed single-linked, however, the shape of the implicit interpolating curve is still far from the circle, from which the randomized points have been collected. Also, various randomization options for the point cloud were evaluated, but none of them led to a satisfactory shape of the interpolating implicit curve.

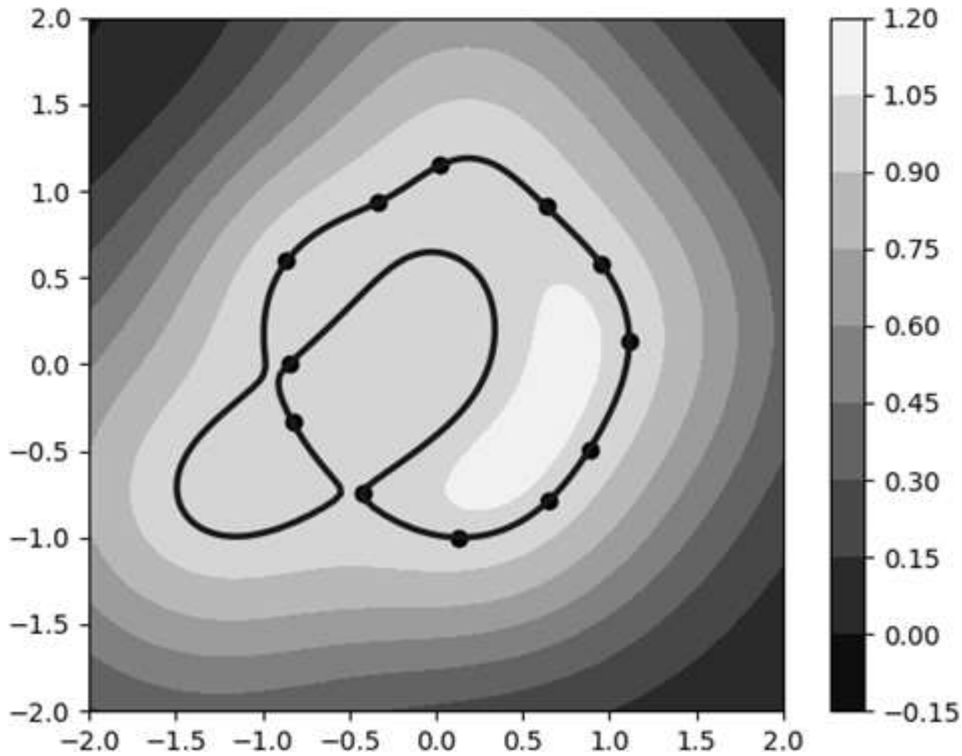

Figure 5. An example of a_i optimization based on the minimized curvature

3. The last target function in scrutiny was the square of the difference between the value of the polynomial and the actual equation of the circle in the implicit form on a predetermined dot grid of 16×16 points uniformly defined at [-1.5, 1.5]×[-1.5, 1.5].

Minimization of this target function actually brought the desired result (Figure 6).

This proves that curve and surface reconstruction from their respective point clouds with an interpolating Gaussian polynomial is possible. However, the very need for an implicit equation for an object similar to the one we should have reconstructed ourselves from the point cloud already negates the value of the method. If the orientation of the surface can be recovered from the Gaussian polynomial, then it could also be recovered from the implicit equation we would use to compute the polynomial in question.

Also, calculating the target function on a dot grid is a computationally demanding task. For instance, computing the a_i coefficients using a classical optimization method for the curve in Figure 6, took about 50 seconds on a computer with an Intel(R) Core(TM) i7-7700HQ CPU @ 2.80GHz processor.

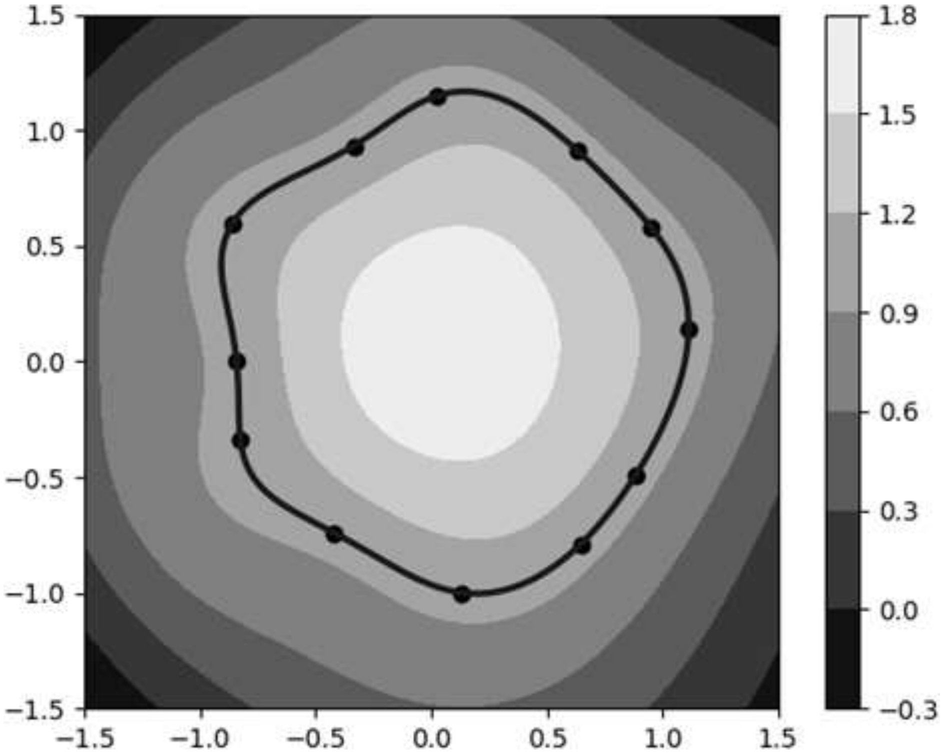

Figure 6. The implicit curve with its difference from the circle equation minimized

Conclusions. Using an interpolating Gaussian polynomial to reconstruct curves and surfaces from their point clouds is possible but the algorithm we fund that does so requires some sort of implicit representation of the goal function preconception. This makes its practical application unfeasible. The algorithm is also computationally demanding even in the 2D case, and even for a small amount of points in a cloud.

However, the very possibility of surface reconstruction, shown by our approach seems promising. We already know that approximating Gaussian polynomials works well in the context of point cloud surface restoration, and now we have demonstrated that the interpolating approach is also plausible. Perhaps, if some hybrid method could be both exact at some given points and approximating for the rest of the point cloud, it could also achieve computational efficiency and model independence our method lacks.

Література

1. Сидоренко Ю.В. Побудова гладких ліній за допомогою параметризованих функцій Гауса. *Прикладна геометрія та інженерна графіка*, 2001. Вип. 69. С. 63-67.
2. Ausheva N.M., Sydorenko Iu.V., Kaleniuk O.S., Kardashov O.V., & Horodetskyi, M. V. Implicit curves and surfaces modeling with pseudogaussian interpolation. *Radio Electronics, Computer Science, Control*, 2025. Issue 1, pp. 30–37. https://doi.org/10.15588/1607-3274-2025-1-3.

3. Аушева , Н., Каленюк , О., Сидоренко , Ю. Побудова замкнених гладких кривих інтерполяційним поліномом гауса. *Сучасні проблеми моделювання*, 2024. Вип. 26, С.21-28. https://doi.org/10.33842/2313125X-2024-26-21-28
4. Salihu, D., Steinbach, E. SGPCR: spherical gaussian point cloud representation and its application to object registration and retrieval. *IEEE/CVF Winter Conference on Applications of Computer Vision (WACV)*, Waikoloa, HI, USA, 2023. pp. 572–581. https://doi.org/10.1109/WACV56688.2023.00064.
5. Zhang K., Luan, F., Wang, Q., Bala, K., Snavely, N. Physg: Inverse rendering with spherical gaussians for physics-based material editing and relighting *IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR)*, 2021. pp 5449 – 5458. https://doi.org/10.48550/arXiv.2104.00674.
6. Tsai, Yu-Ti., Shih, Zen-Ch. All-frequency precomputed radiance transfer using spherical radial basis functions and clustered tensor approximation. *ACM Transactions on Graphics (TOG)*, 2006. 25(3), pp. 967 – 976. https://doi.org/10.1145/1141911.1141981.

ПОБУДОВА ЗАМКНЕНИХ ГЛАДКИХ КРИВИХ ІНТЕРПОЛЯЦІЙНИМ ПОЛІНОМОМ ГАУСА

Тарнавський Ю.А., Михайлова І.Ю., Каленюк О.С.

У статті розглядаються перспективи використання специфічного типу гаусової поліноміальної функції для реконструкції неперервної та регулярної поверхні її хмарою точок.

Відновлення кривих і поверхонь за їх хмарою точок є важливою підпроблемою для встановлення технічних поверхонь за їх формою за допомогою 3D-сканування. Також аналогічна проблема виникає при перетворенні поверхонь з граничного представлення в трикутну сітку. Для останніх критично важливо відновити орієнтацію поверхні, якщо вихідні дані не містять такої інформації, або містять неправдиві відомості.

Гаусівський інтерполяційний многочлен прямує йде до нуля, коли його аргумент прямує до плюс- і мінус-нескінченності при будь-якій конфігурації інтерполяційних точок. Таким чином, якщо в тривимірному просторі ми визначаємо поверхню облака точок, як ізоповерхню імпліцитної функції з позитивним значенням, наприклад, 1, і вважатимемо зовнішньою орієнтацію поверхні простір, в якому значення імпліцитної функції менші за одиницю, то поверхня, побудована узагальненим інтерполяційним многочленом Гауса, завжди буде орієнтованою правильно. Тобто простір, обмежений поверхнею, завжди буде замкненим і це замикання буде містити внутрішню сторону поверхні. При тому, сама ізоповерхня буде неперервною за всіма частковими похідними а значить і регулярною, адже Гаусіани, з яких складається поліном, диференцюються нескінченно.

Незважаючи на ці корисні властивості, саме інтерполяційний многочлен Гауса дотепер при відновленні кривих на основі хмари точок не використовувався. Отже метою даного дослідження є оцінка перспектив практичного застосування інтерполяції Гауса в контексті відновлення кривих і поверхонь за їх хмарами точок.

Адже мета такого застосування полягає не стільки в отриманні правильних диференціальних властивостей результуючих кривих і поверхонь, а у встановленні їх топологічних властивостей, а саме зв'язності, і результуючих форм. Форма кривої або поверхні, отримана шляхом інтерполяції Гауса, повинна певним чином відповідати передбачуваній поверхні або кривій, яка преставлена хмарою точок.

Ключові слова: інтерполяція, многочлен Гауса, регулярні поверхні, хмара точок, математична оптимізація, геометричне моделювання, моделювання.

References

1. Sydorenko Iu.V. (2001) Pobudova gladkykh linij za dopomogoyu parametryzovanikh funkcij Gausa. *Prykladna geometriya ta inzhenerna grafika*. Issues 69, 63-67. [in Ukrainian].
2. Ausheva N. M., Sydorenko Iu. V., Kaleniuk O. S., Kardashov O. V., & Horodetskyi M. V. (2025). Implicit Curves And Surfaces Modeling With Pseudogaussian Interpolation. *Radio Electronics, Computer Science, Control*. Issues 1, 30–37. https://doi.org/10.15588/1607-3274-2025-1-3
3. Ausheva N. M., Kaleniuk O. S., Sydorenko Iu. V. (2024) Closed smooth curves construction with the gaussian interpolating polynomial. *Suchasni problemy modeliuvannia*. Issues 26, 21–28. https://doi.org/10.33842/2313125X-2024-26-21-28.
4. Salihu, D., Steinbach, E. (2023) SGPCR: Spherical Gaussian Point Cloud Representation and its Application to Object Registration and Retrieval *IEEE/CVF Winter Conference on Applications of Computer Vision (WACV),* Waikoloa, HI, USA, 572–581, https://doi.org/10.1109/WACV56688.2023.00064
5. Zhang K., Luan, F., Wang, Q., Bala, K., Snavely, N. (2021) Physg: Inverse rendering with spherical gaussians for physics-based material editing and relighting IE*EE/CVF Conference on Computer Vision and Pattern Recognition (CVPR)*, 5449 – 5458. https://doi.org/10.48550/arXiv.2104.00674.
6. Tsai, Yu-Ti., Shih, Zen-Ch. (2006) All-frequency precomputed radiance transfer using spherical radial basis functions and clustered tensor approximation. *ACM Transactions on Graphics (TOG),* 25 (3), 967 – 976. https://doi.org/10.1145/1141911.1141981.